

ARITHMETIC LAPLACIANS

ALEXANDRU BUIUM AND SANTIAGO R. SIMANCA

ABSTRACT. We develop an arithmetic analogue of elliptic partial differential equations. The rôle of the space coordinates is played by a family of primes, and that of the space derivatives along the various primes are played by corresponding Fermat quotient operators subjected to certain commutation relations. This leads to arithmetic linear partial differential equations on algebraic groups that are analogues of certain operators in analysis constructed from Laplacians. We classify all such equations on one dimensional groups, and analyze their spaces of solutions.

1. INTRODUCTION

1.1. Main concepts and results. In a series of articles (beginning with [5]), the first author developed an arithmetic analogue of ordinary differential equations (ODEs); cf. [9] for an account of this theory. The aim of the present work is to extend some of this theory to the partial differential case by considering arithmetic analogues of elliptic partial differential equations (PDEs). A different extension of the ODE theory, in the direction of parabolic and hyperbolic PDEs, was developed by the authors in [10, 11]. The work here relies on the ODE theory but is independent of [10, 11].

Before explaining the main results of this paper let us put our present work in perspective. The general aim of this paper and of [10, 11] is to pass from one variable to several variables. The independent variables we have in mind are of two types, the “geometric independent variables,” whose number we denote by an integer $d_1 \geq 0$, and the “arithmetic independent variables,” whose number we denote by the integer $d_2 \geq 1$. We would like to extend the ODE theory to a PDE theory in $d_1 + d_2$ independent variables. We explain this in some detail next. For convenience, we introduce a third integer parameter $d_3 \geq 1$ to denote the number of “dependent variables” that may arise in pursuing the said extension.

The ODE theory in [5, 9] corresponds to the case of $0 + 1$ independent variables, where the one independent arithmetic variable is represented by a prime integer p . The role of the derivative with respect to p is played by a Fermat quotient operator δ_p that on integers $a \in \mathbb{Z}$ acts as

$$\delta_p a = \frac{a - a^p}{p}.$$

If we view this theory in a manner analogous to particle mechanics, as in [9], then the prime p is the analogue of “time,” and d_3 is the dimension of the “configuration space.” If we pursue the analogy with field theory instead, then p is the analogue of a “space” variable and d_3 is the dimension of the “space of internal states.”

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The PDE theory in [10, 11] corresponds to the case of $1 + 1$ independent variables. The one geometric variable, q , is viewed as the exponential of “time,” and the one arithmetic variable, viewed as the analogue of space, is represented, again, by a prime p . The derivative in the geometric direction is given by the usual derivation $\delta_q = qd/dq$, while the role of the derivative with respect to p is again played by a Fermat quotient operator δ_p . The main purpose of [10] was to determine all linear PDEs on the algebraic groups of dimension $d_3 = 1$ (precisely, on the group schemes \mathbb{G}_a , \mathbb{G}_m , E , where E is any elliptic curve), and to study the solutions of all such arithmetic linear PDEs. In line with [5, 9], these arithmetic linear PDEs are referred to as $\{\delta_p, \delta_q\}$ -characters. They can be viewed as arithmetic analogues of evolution equations in analysis; for \mathbb{G}_m , the basic examples are analogues of convection equations whereas for elliptic curves, the basic examples are analogues of convection, heat, and wave equations. In [11], the one dimensional groups were replaced by modular curves. The resulting equations now can be viewed as arithmetic analogues of convection and heat equations. (Wave equations do not have analogues in the modular case.)

In this article, we develop an arithmetic PDE theory in $0 + d_2$ independent variables, where $d_2 \geq 2$, integer which for simplicity we now simply denote by d . We view the new equations as independent of the geometric time, but the analogue of the space coordinates is now a finite set $\mathcal{P} = \{p_1, \dots, p_d\}$ of primes. The space derivatives in the various directions given by the various primes correspond to Fermat quotient operators $\delta_{\mathcal{P}} = \{\delta_{p_1}, \dots, \delta_{p_d}\}$ that are subjected to certain commutation relations.

Let us denote by $\mathbb{Z}_{(\mathcal{P})}$ the ring of fractions of \mathbb{Z} with respect to the multiplicative system of all integers coprime to the primes in \mathcal{P} . Our first task is to define “arithmetic PDEs” with respect to $\delta_{\mathcal{P}}$ on a smooth scheme X over $\mathbb{Z}_{(\mathcal{P})}$ of relative dimension $d_3 \geq 1$, which will not necessarily be of a linear nature. In the applications d_3 will be taken to be 1. We proceed to explain the main ideas when in the scheme X is affine.

Following the case of classical derivations [19, 4], we would like to define our PDEs as “functions on jet spaces.” Thus, we first construct *arithmetic jet spaces* $\mathcal{J}_{\mathcal{P}}^r(X)$ of order $r = (r_1, \dots, r_d) \in \mathbb{Z}_+^d$ (that is to say, of order r_k with respect to δ_{p_k} for all k) by analogy with the case of derivations. When dealing with a single prime p , a p -adically completed version of these spaces was introduced and thoroughly analyzed by the first author in [5, 6], and in a number of subsequent papers; cf. [9] for an exposition of this theory. In the case where the number of primes d is strictly greater than 1, our arithmetic jet spaces were also independently introduced in a recent preprint by Borger [2], where they were denoted by $W_{r*}(X)$.

As it turns out, in most cases the ring A of global functions on a jet space $\mathcal{J}_{\mathcal{P}}^r(X)$ does not contain “interesting elements” (e.g. functions that qualify as analogues of linear differential operators). However, we show that, for each of the primes p_k in \mathcal{P} , we can construct “interesting” formal functions $f_k \in \widehat{A^{p_k}}$, where $\widehat{A^{p_k}}$ is the p_k -adic completion of A . Further, we would then like to be able to say when certain families f_1, \dots, f_d of such formal functions “can be glued” together. Since the “domains of definition” of these f_k s are disjoint (each f_k is defined on a “vertical tubular neighborhood” of $\text{Spec } A/p_k A$), we need an “indirect” approach to the gluing of these formal functions. We propose here to do this “gluing” by some sort of “analytic continuation” between different primes, and proceed as follows.

We first fix a $\mathbb{Z}_{(\mathcal{P})}$ -point P of X . This point can be lifted canonically to a $\mathbb{Z}_{(\mathcal{P})}$ -point P^r of $\mathcal{J}_p^r(X)$, and we still denote by $P^r \subset A$ the ideal of the image of P^r . Then, we declare that a family $f = (f_k) \in \prod_{k=1}^d A^{\widehat{p_k}}$ can be *analytically continued along P* if there exists $f_0 \in A^{\widehat{P^r}}$ such that, for each $k = 1, \dots, d$, the images of f_0 and f_k in $A^{\widehat{(p_k, P^r)}}$ coincide. Here, the superscripts “ $\widehat{}$ ” mean completions with respect to the corresponding ideals. Intuitively, f_0 can be viewed as a formal function defined on a “horizontal tubular neighborhood” of the canonical lift to the jet space of our fixed point P of X . This horizontal tubular neighborhood meets each of the vertical tubular neighborhoods transversally in one point, hence, in particular, the union of the vertical tubular neighborhoods with the horizontal one is connected and, actually, it “looks like a tree” so is morally “simply connected;” this justifies the terminology of *analytic continuation*. Families f that can be analytically continued along P will be referred to as $\delta_{\mathcal{P}}$ -functions, and will be viewed as arithmetic (non-linear) PDEs on X . The analytic continuation concept really depends on the choice of P . However, we will provide a description of this dependence in the special cases under consideration.

This can all be generalized to the case when X is not affine, though some extra difficulties have to be overcome. In the case when X is a group scheme G and P is the identity, the “additive” $\delta_{\mathcal{P}}$ -functions will be called $\delta_{\mathcal{P}}$ -characters; morally these play the role of the linear arithmetic PDEs on G . Our main result is the determination of all $\delta_{\mathcal{P}}$ -characters on $G = \mathbb{G}_a$, $G = \mathbb{G}_m$, or $G = E$. (Here E is, again, an elliptic curve.) Indeed let $e_k = (0, \dots, 1, \dots, 0) \in \mathbb{Z}_+^d$, with 1 on the k th place, and let $e = e_1 + \dots + e_d = (1, \dots, 1)$. Then, in each of the 3 cases $G = \mathbb{G}_a$, $G = \mathbb{G}_m$, or $G = E$, the spaces of $\delta_{\mathcal{P}}$ -characters are generated by a fundamental $\delta_{\mathcal{P}}$ -character of order $se = (s, \dots, s)$, where $s = 0, 1, 2$, respectively. These fundamental $\delta_{\mathcal{P}}$ -character will be denoted by $\psi_G = \psi_a^0$, $\psi_G = \psi_m^s$, and $\psi_G = \psi_E^{2e}$, respectively. (As a general convention, upper indices for PDEs stand for order.) In each of the 3 cases we have decompositions

$$(1) \quad \psi_G = (\bar{\psi}_{\bar{p}_k} \psi_{p_k})$$

where ψ_{p_k} come from δ_{p_k} -characters of G of order $s = 0, 1, 2$, respectively (in the sense of the arithmetic ODE theory in [5]) and $\bar{\psi}_{\bar{p}_k}$ come from $\delta_{\bar{p}_k}$ -characters of \mathbb{G}_a , where \bar{p}_k is the set $\mathcal{P} \setminus \{p_k\}$. Each $\bar{\psi}_{\bar{p}_k}$ is, loosely speaking, the product of the “symbols” of all ψ_{p_l} with $l \neq k$. These “symbols” are related to the Euler factors of certain corresponding L -functions. (The case $G = \mathbb{G}_a$ does not actually require L -functions; for $G = \mathbb{G}_m$ and $G = E$ the L -functions are the Riemann zeta function and the Hasse-Weil L -function respectively.) The equality (1) will be viewed as a *Dirac decomposition* of ψ_G ; cf. the next subsection for analytic analogues of this. Since one uses [5], the construction of ψ_G is global. The fact that ψ_G generates the space of $\delta_{\mathcal{P}}$ -characters will be proved by a purely local method based on results of Honda, Hill, and Hazewinkel; cf [14]. We will also show, in each of the 3 cases, that there are plenty of $\delta_{\mathcal{P}}$ -characters that can be analytically continued along *any* of the $\mathbb{Z}_{(\mathcal{P})}$ -points of G (although this is not true in general about ψ_G itself). Finally we will also show, in each of the 3 cases, that the solution group of $\psi_G = 0$ in appropriate cyclotomic rings consists of the torsion points of G with values in those rings; this should be viewed as an arithmetic analogue of an analytic statement that says that the only harmonic functions (in an appropriate setting) are the constants; cf. the discussion in the next subsection.

1.2. Analytic analogues. When $d = 2$ (and $G = \mathbb{G}_m$ or $G = E$), our fundamental $\delta_{\mathcal{P}}$ -characters ψ_G are arithmetic analogues of operators in analysis that are related to Laplacians. We explain here this analogy.

Our fundamental $\delta_{\mathcal{P}}$ -character ψ_m^e on \mathbb{G}_m should be viewed as an arithmetic analogue of the partial differential operator

$$(2) \quad \begin{array}{ccc} \psi_{z\bar{z}}^e : C^\infty(D, \mathbb{C}^\times) & \rightarrow & C^\infty(D, \mathbb{C}) \\ u & \mapsto & \psi_{z\bar{z}}^e(u) := \frac{1}{4} \Delta \log u = \partial_z \partial_{\bar{z}} \log u, \end{array}$$

where $D \subset \mathbb{C}$ is a domain, $z = x + iy$ is the complex coordinate on D , and $\Delta = \partial_x^2 + \partial_y^2$ is the Euclidean Laplacian. (Here $\partial_x, \partial_y, \partial_z, \partial_{\bar{z}}$ are the corresponding partial derivative operators.) Note that, like our arithmetic ψ_m^e , the operator $\psi_{z\bar{z}}^e$ is a group homomorphism and has a “Dirac decomposition:”

$$\psi_{z\bar{z}}^e(u) = \partial_z \left(\frac{\partial_{\bar{z}} u}{u} \right) = \partial_{\bar{z}} \left(\frac{\partial_z u}{u} \right)$$

which is analogous to the decomposition (1). Note that if we equip D and \mathbb{C}^\times with their usual complex structure, and we equip \mathbb{C}^\times with the conformal metric $\frac{dz d\bar{z}}{z\bar{z}}$ then $\psi_{z\bar{z}}^e$ is an instance of the operator in [17], p. 124 appearing in the theory of harmonic maps between Riemann surfaces. Also recall from [17], p. 63, that for real positive u we have $\Delta \log u = -K_u \cdot u^2$, where K_u is the curvature of the conformal metric $g = u^2 dz d\bar{z}$. If instead of D we take an arbitrary Riemann surface Σ then $c_1(g) := -\frac{1}{4\pi i} K_u \cdot u^2 dz \wedge d\bar{z}$ is the Chern form of the metric g . So our character ψ_m^e can also be viewed as an arithmetic analogue of the operator

$$(3) \quad \begin{array}{ccc} \psi_\Sigma^e : \{\text{conformal metrics on } \Sigma\} & \rightarrow & \{(1, 1)\text{-forms on } \Sigma\} \\ g & \mapsto & \psi_\Sigma^e(g) = c_1(g) \end{array}.$$

Similarly, our fundamental $\delta_{\mathcal{P}}$ -character ψ_E^{2e} on an elliptic curve E can be viewed as an arithmetic analogue of an analytic operator that we now describe. Let D be a domain contained in the upper half plane $\{\tau = x + iy \in \mathbb{C} ; y > 0\}$. Let $E = (D \times \mathbb{C}) / \sim$ where $(\tau_1, z_1) \sim (\tau_2, z_2)$ iff $\tau_1 = \tau_2 =: \tau$ and $z_2 - z_1 \in \mathbb{Z}\tau + \mathbb{Z}$. Let $\pi : E \rightarrow D$ be the canonical projection (so E is the universal elliptic curve over D). Let $C_\pi^\infty(D, E)$ the set of all maps $\sigma \in C^\infty(D, E)$ such that $\pi \circ \sigma = 1_D$. Let $\log_E : E \rightarrow \mathbb{C}$ be the (multivalued) map obtained by composing the (multivalued) inverse of the canonical projection $D \times \mathbb{C} \rightarrow E$ with the second projection $D \times \mathbb{C} \rightarrow \mathbb{C}$. Then our arithmetic ψ_E^{2e} can be viewed as an analogue of the order 4 map

$$(4) \quad \begin{array}{ccc} \psi_{\tau\bar{\tau}}^{2e} : C_\pi^\infty(D, E) & \rightarrow & C^\infty(D, \mathbb{C}) \\ u & \mapsto & \psi_{\tau\bar{\tau}}^{2e}(u) := \frac{1}{16} \Delta \Delta \log_E u = \partial_\tau^2 \partial_{\bar{\tau}}^2 \log_E u, \end{array}$$

where $\Delta = \partial_x^2 + \partial_y^2$ is the Euclidean Laplacian. (In order to see that this map is well defined, notice that if $\varphi : D \rightarrow \mathbb{C}$ is a smooth map satisfying $\varphi(\tau) \in \mathbb{Z}\tau + \mathbb{Z}$ for all $\tau \in D$, then there exist $m, n \in \mathbb{Z}$ such that $\varphi(\tau) = m\tau + n$ for all $\tau \in D$ and hence φ is in the kernel of $\partial_\tau^2 \partial_{\bar{\tau}}^2$.) The map $\psi_{\tau\bar{\tau}}^{2e}$ is a group homomorphism, and has a “Dirac decomposition”

$$(5) \quad \psi_{\tau\bar{\tau}}^{2e}(u) = \partial_{\bar{\tau}}^2(\mu_\tau(u)) = \partial_\tau^2(\mu_{\bar{\tau}}(u)),$$

where

$$(6) \quad \mu_\tau(u) = \partial_\tau^2 \log_E u, \quad \mu_{\bar{\tau}}(u) = \partial_{\bar{\tau}}^2 \log_E u.$$

This is again analogous to (1). The map μ_τ is the analytic expression of the Manin map in [21], and is well defined because any map $\varphi(\tau) = m\tau + n$ ($m, n \in \mathbb{Z}$) is in the kernel of ∂_τ^2 . Also, $\mu_{\bar{\tau}}$ is well defined because any φ as above is in the kernel of $\partial_{\bar{\tau}}^2$. Note also that one could consider, in this analytic setting, the order 2 map

$$(7) \quad \begin{aligned} \psi_{\tau\bar{\tau}}^e : C_\pi^\infty(D, E) &\rightarrow C^\infty(D, \mathbb{C}) \\ u &\mapsto \psi_{\tau\bar{\tau}}^e(u) := \frac{1}{4} \Delta \log_E u = \partial_\tau \partial_{\bar{\tau}} \log_E u. \end{aligned}$$

(This map is well defined because any $\varphi(\tau) = m\tau + n$ is in the kernel of $\partial_\tau \partial_{\bar{\tau}}$.) The map $\psi_{\tau\bar{\tau}}^2$ does not seem to have, however, a natural “Dirac decomposition” and it has no arithmetic analogue. (The “obvious” candidate for a “Dirac decomposition” for $\psi_{\tau\bar{\tau}}^e$, analogous to (5), fails to make sense because the operator $u \mapsto \partial_\tau \log_E u$ is not well defined.)

Let us recall that the Manin map [21] is an order 2 ODE map introduced by Manin for abelian varieties, in order to prove the Mordell conjecture over function fields. A different construction of this map was given in [4]. The construction in [4] has an arithmetic analogue which was introduced in [5], giving rise to an order 2 ODE *arithmetic Manin map* $\psi_{p_k}^2$. Our arithmetic PDE ψ_E^{2e} has a Dirac decomposition involving the operators $\psi_{p_k}^2$.

Instead of the family $E \rightarrow D$ above we could as well consider a fixed elliptic curve $E_0 = \mathbb{C}/(\mathbb{Z}\tau_0 + \mathbb{Z})$. If $\log_{E_0} : E_0 \rightarrow \mathbb{C}$ is the (multivalued) inverse of the canonical projection $\mathbb{C} \rightarrow E_0$, and D is a domain in the complex plane (with coordinate $w = x + iy$) then we have a well defined map

$$(8) \quad \begin{aligned} \psi_{w\bar{w}}^e : C^\infty(D, E_0) &\rightarrow C^\infty(D, \mathbb{C}) \\ u &\mapsto \psi_{w\bar{w}}^e(u) := \frac{1}{4} \Delta \log_{E_0} u = \partial_w \partial_{\bar{w}} \log_{E_0} u, \end{aligned}$$

admitting an obvious “Dirac decomposition.” There is an arithmetic analogue of this map which we are not going to discuss in the present paper because such a discussion would require developing our theory in a setting slightly more general than the one we have adopted here. In this more general setting, \mathbb{Q} needs to be replaced by a number field F containing an imaginary quadratic field K . Then the analogue of E_0 is an elliptic curve over F with complex multiplication by K , the primes in \mathcal{P} are replaced by primes of F of degree one, where E_0 has ordinary good reduction, and the L -function appearing is the L -function associated to the corresponding Grossencharacter.

1.3. p -adic notation. Given a prime $p \in \mathbb{Z}$, we set

$$\begin{aligned} \mathbb{Z}_{(p)} &= \text{localization of } \mathbb{Z} \text{ at } (p), \\ \mathbb{Z}_p &= \text{ring of } p\text{-adic integers}, \end{aligned}$$

notation that we use throughout the paper. Given a family $\mathcal{P} = \{p_1, \dots, p_d\}$ of primes in \mathbb{Z} , we set

$$\mathbb{Z}_{(\mathcal{P})} = \bigcap_{k=1}^d \mathbb{Z}_{(p_k)}.$$

For each $k = 1, \dots, d$, we set

$$\bar{p}_k = \mathcal{P} \setminus \{p_k\} = \{p_1, \dots, p_{k-1}, p_{k+1}, \dots, p_d\}.$$

If A is a ring and I is an ideal, we let $A^{\hat{I}}$ denote the completion of A with respect to I . If the ideal I is generated by a family of elements a , we shall write $A^{\hat{a}}$ instead

of $A^{\widehat{I}}$. In particular, for any prime $p \in \mathbb{Z}$, we have that $A^{\widehat{p}}$ is the p -adic completion of A .

If X is any Noetherian scheme, we denote by $X^{\widehat{p}}$ the p -adic completion of X . For any ring A and family of primes $\mathcal{P} = \{p_1, \dots, p_d\}$ as above, we set

$$A_{\mathcal{P}} = \prod_{k=1}^d A^{\widehat{p_k}}.$$

1.4. Structure of the paper. In §2, we present our main concepts, in particular, the notions of $\delta_{\mathcal{P}}$ -jet spaces and $\delta_{\mathcal{P}}$ -characters. In §3, we state and prove our main results determining all the $\delta_{\mathcal{P}}$ -characters on one dimensional groups. In §4 we present a number of remarks and open questions.

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2. MAIN CONCEPTS

2.1. $\delta_{\mathcal{P}}$ -rings. For any prime $p \in \mathbb{Z}$, we consider the polynomial

$$C_p(X, Y) := \frac{X^p + Y^p - (X + Y)^p}{p} \in \mathbb{Z}[X, Y].$$

Let A be a ring and B an A -algebra. For $a \in A$, we denote the element $a \cdot 1_B \in B$ by a also.

Definition 2.1. A map $\delta_p : A \rightarrow B$ is a p -derivation if

$$(9) \quad \begin{aligned} \delta_p(a + b) &= \delta_p a + \delta_p b + C_p(a, b), \\ \delta_p(ab) &= a^p \delta_p b + b^p \delta_p a + p \delta_p a \delta_p b, \end{aligned}$$

for all $a, b \in A$. □

If δ_p is a p -derivation, then the map

$$(10) \quad \begin{array}{ccc} A & \xrightarrow{\phi_p} & B \\ a & \mapsto & \phi_p(a) := a^p + p \delta_p a \end{array}$$

is a ring homomorphism, and we have that

$$(11) \quad \phi_p(a) \equiv a^p \pmod{p}$$

for all $a \in A$.

Conversely, let $\phi_p : A \rightarrow B$ be a ring homomorphism satisfying (11), which we refer to as a *lift of Frobenius* homomorphism. Let us assume in addition that p is a non-zero divisor in B . Then ϕ_p has the form (10) for a unique p -derivation δ_p ,

$$\delta_p a = \frac{\phi_p(a) - a^p}{p},$$

and we have that

$$\phi_p \delta_p a = \delta_p \phi_p a$$

for all $a \in A$. We say that δ_p and ϕ_p are *associated* to each other.

In particular, the ring $A = \mathbb{Z}$ has a unique p -derivation $\delta_p : \mathbb{Z} \rightarrow \mathbb{Z}$, given by

$$\delta_p a := \frac{a - a^p}{p}.$$

If p_1 and p_2 are two distinct primes in \mathbb{Z} , we now consider the polynomial C_{p_1, p_2} in the ring $\mathbb{Z}[X_0, X_1, X_2]$ defined by

$$(12) \quad C_{p_1, p_2}(X_0, X_1, X_2) := \frac{C_{p_2}(X_0^{p_1}, p_1 X_1)}{p_1} - \frac{C_{p_1}(X_0^{p_2}, p_2 X_2)}{p_2} - \frac{\delta_{p_1} p_2}{p_2} X_2^{p_1} + \frac{\delta_{p_2} p_1}{p_1} X_1^{p_2}.$$

Let us notice that

$$(13) \quad C_{p_1, p_2} \in (X_0, X_1, X_2)^{\min\{p_1, p_2\}}.$$

Definition 2.2. Let $\mathcal{P} = \{p_1, \dots, p_d\}$ be a finite set of primes in \mathbb{Z} . A $\delta_{\mathcal{P}}$ -ring is a ring A equipped with p_k -derivations $\delta_{p_k} : A \rightarrow A$, $k = 1, \dots, d$, such that

$$(14) \quad \delta_{p_k} \delta_{p_l} a - \delta_{p_l} \delta_{p_k} a = C_{p_k, p_l}(a, \delta_{p_k} a, \delta_{p_l} a)$$

for all $a \in A$, $k, l = 1, \dots, d$. A *homomorphism of $\delta_{\mathcal{P}}$ -rings* A and B is a homomorphism of rings $\varphi : A \rightarrow B$ that commutes with the p_k -derivations in A and B , respectively. \square

If ϕ_{p_k} is the homomorphism (10) associated to δ_{p_k} , condition (14) implies that

$$(15) \quad \phi_{p_k} \phi_{p_l}(a) = \phi_{p_l} \phi_{p_k}(a)$$

for all $a \in A$. Conversely, if the commutation relations (15) hold, and the p_k s are non-zero divisors in A , then conditions (14) hold, and we have that

$$\phi_{p_k} \delta_{p_l} a = \delta_{p_l} \phi_{p_k} a$$

for all $a \in A$.

Remark 2.3. The concept of a “lift of Frobenius” homomorphism that is the basis for the definitions given above, is classical and goes back to work of Frobenius, Chebotarev, and Artin on number fields. It plays a key role in the theory of Witt vectors (in particular, in crystalline cohomology), and it resurfaced in the context of K -theory through Grothendieck’s concept of lambda ring; cf. [13, 14, 23, 18, 3] for this circle of ideas. This concept was taken in [5, 8, 9] as the starting point for developing an arithmetic analogue of ordinary differential equations on commutative algebraic groups and on moduli spaces (such as modular curves, Shimura curves, and Siegel modular varieties). In this article, we aim at extending these ideas to the partial differential case (at least for one dimensional algebraic groups). \square

Remark 2.4. Let A be a $\delta_{\mathcal{P}}$ -ring. Then for all k , the p_k -adic completions $\widehat{A^{p_k}}$ are $\delta_{\mathcal{P}}$ -rings in a natural way. For ϕ_{p_l} extends to $\widehat{A^{p_k}}$ by continuity. The condition $\phi_{p_k} a \equiv a^{p_k} \pmod{p_k}$ in $\widehat{A^{p_k}}$ holds by continuity, while the condition $\phi_{p_l} a \equiv a^{p_l} \pmod{p_l}$ in $\widehat{A^{p_k}}$ holds because p_l is invertible in $\widehat{A^{p_k}}$. \square

We let $\mathbb{Z}_+ = \{0, 1, 2, 3, \dots\}$, and let \mathbb{Z}_+^d be given the product order, defined by declaring that

$$r = (r_1, \dots, r_d) \leq r' = (r'_1, \dots, r'_d)$$

if $r_k \leq r'_k$ for all $k = 1, \dots, d$. The weight $|r|$ of the d -tuple r is defined by $|r| = \sum_{j=1}^d r_j$. Finally, we let e_k be the element of \mathbb{Z}_+^d all of whose components are zero except the k -th, which is 1.

In more generality, we have the following:

Definition 2.5. A $\delta_{\mathcal{P}}$ -prolongation system $A^* = (A^r)$ is an inductive system of rings A^r indexed by $r \in \mathbb{Z}_+^d$, provided with transition maps $\varphi_{rr'} : A^r \rightarrow A^{r'}$ for any pair of indices r, r' such that $r \leq r'$, and equipped with p_k -derivations

$$\delta_{p_k} : A^r \rightarrow A^{r+e_k},$$

$k = 1, \dots, d$, such that (14) holds for all k, l , and such that

$$\varphi_{r+e_k, r'+e_k} \circ \delta_{p_k} = \delta_{p_k} \circ \varphi_{rr'} : A^r \rightarrow A^{r'+e_k}$$

for all $r \leq r'$ and all k . A morphism of prolongation systems $A^* \rightarrow B^*$ is a system of ring homomorphisms $u^r : A^r \rightarrow B^r$ that commute with the φ s and the δ s of A^* and B^* , respectively. \square

Any $\delta_{\mathcal{P}}$ -ring A induces a $\delta_{\mathcal{P}}$ -prolongation system A^* where $A^r = A$ for all r and $\varphi = \text{identity}$. If A is a $\delta_{\mathcal{P}}$ -ring and A^* is the associated $\delta_{\mathcal{P}}$ -prolongation system, we say that a $\delta_{\mathcal{P}}$ -prolongation system B^* is a $\delta_{\mathcal{P}}$ -prolongation system over A if it is equipped with a morphism $A^* \rightarrow B^*$. We have a natural concept of morphism of $\delta_{\mathcal{P}}$ -prolongation systems over A .

Example 2.6. Let $S \subset \mathbb{Z}$ be a multiplicative system of integers coprime to p_1, \dots, p_d , and let $\mathbb{Z}_S = S^{-1}\mathbb{Z}$ be the ring of fractions. Given an integer m coprime to p_1, \dots, p_d , let $\zeta_m = \exp\left(\frac{2\pi\sqrt{-1}}{m}\right)$. We consider the ring $A = \mathbb{Z}_S[\zeta_m]$. Let $\phi_{p_1}, \dots, \phi_{p_d} \in G(\mathbb{Q}(\zeta_m)/\mathbb{Q}) = (\mathbb{Z}/m\mathbb{Z})^\times$ be the Galois elements corresponding to the classes of p_1, \dots, p_d , respectively. Then $\phi_{p_k}(a) \equiv a^{p_k} \pmod{p_k}$ for $k = 1, \dots, d$ and $a \in A$. Thus, A is a $\delta_{\mathcal{P}}$ -ring with respect to the p_k -derivations δ_{p_k} associated to the ϕ_{p_k} s. \square

From now on, A_0 will denote a $\delta_{\mathcal{P}}$ -ring that is an integral Noetherian domain of characteristic zero, with fraction field K_0 . Notice that K_0 has a naturally induced structure of $\delta_{\mathcal{P}}$ -ring.

Example 2.7. Let $A = A_0[[q]]$ be the power series ring in the indeterminate q . We let

$$\phi_{p_k} : A \rightarrow A$$

$1 \leq k \leq d$ be the family of homomorphisms defined by

$$\phi_{p_k}\left(\sum c_n q^n\right) = \sum \phi_{p_k}(c_n) q^{np_k}.$$

$k = 1, \dots, d$. Then, as before A is a $\delta_{\mathcal{P}}$ -ring with respect to the p_k -derivations δ_{p_k} associated to the ϕ_{p_k} s.

Let x be an n -tuple of variables. We consider n -tuples of variables x_i indexed by vectors $i = (i_1, \dots, i_d)$ in \mathbb{Z}_+^d . We set $x = x_{(0, \dots, 0)}$, $\delta_{\mathcal{P}}^i = \delta_{p_1}^{i_1} \dots \delta_{p_d}^{i_d}$, and $\phi_{\mathcal{P}}^i = \phi_{p_1}^{i_1} \dots \phi_{p_d}^{i_d}$.

Let $K_0\{x\}$ be the ring of polynomials

$$K_0\{x\} := K_0[x_i : i \in \mathbb{Z}_+^d]$$

with K_0 -coefficients in the variables x_i . We extend the homomorphisms $\phi_{p_k} : A_0 \rightarrow A_0$ to ring endomorphisms $\phi_{p_k} : K_0\{x\} \rightarrow K_0\{x\}$ by the formulae

$$\phi_{p_k}(x_i) = x_{i+e_k},$$

so that $x_i = \phi_{\mathcal{P}}^i x$ for all i . Clearly $\phi_{p_k} \phi_{p_l}(a) = \phi_{p_l} \phi_{p_k}(a)$ for all $a \in K_0\{x\}$, and all k, l . If we consider the p_k -derivations $\delta_{p_k} : K_0\{x\} \rightarrow K_0\{x\}$ associated to the

ϕ_{p_k} s then $K_0\{x\}$ is a $\delta_{\mathcal{P}}$ -ring. Notice that $K_0\{x\}$ is generated as a K_0 -algebra by the elements $\delta_{\mathcal{P}}^i x$, $i \in \mathbb{Z}_+^d$:

$$K_0\{x\} = K_0[\delta_{\mathcal{P}}^i x : i \in \mathbb{Z}_+^d].$$

Example 2.8. We define the ring of $\delta_{\mathcal{P}}$ -polynomials $A_0\{x\}$ to be the A_0 -subalgebra of $K_0\{x\}$ generated by all the elements $\delta_{\mathcal{P}}^i x$:

$$A_0\{x\} := A_0[\delta_{\mathcal{P}}^i x : i \in \mathbb{Z}_+^d].$$

Notice that $A_0\{x\}$ is strictly larger than the ring $A_0[x_i : i \in \mathbb{Z}_+^d]$. And notice also that the family $\{\delta_{\mathcal{P}}^i : i \in \mathbb{Z}_+^d\}$ is algebraically independent over A_0 , so $A_0\{x\}$ is a ring of polynomials in the “variables” $\delta_{\mathcal{P}}^i x$.

The ring $A_0\{x\}$ has a natural structure of $\delta_{\mathcal{P}}$ -ring due to the following:

Lemma 2.9. *We have that $\delta_{p_k} A_0\{x\} \subset A_0\{x\}$ for $k = 1, \dots, d$.*

Proof. By (2.1), the sets $S_k := \{a \in K_0\{x\} : \delta_{p_k} a \in A_0\{x\}\}$ are A_0 -subalgebras of $K_0\{x\}$, so it is enough to show that $\delta_{\mathcal{P}}^i x \in S_k$ for all i and k . We use the commutation relations (14) to check this by induction on $(i, k) \in \mathbb{Z}_+^d \times \mathbb{Z}_+$ with respect to the lexicographic order. \square

Example 2.10. Proceeding exactly as in Example 2.8, the system

$$A_0[\delta_{\mathcal{P}}^i x : i \leq r]$$

has a natural structure of $\delta_{\mathcal{P}}$ -prolongation system. \square

Example 2.11. Let T be a tuple of indeterminates, and let

$$A^r = A_0[[\delta_{\mathcal{P}}^i T : i \leq r]].$$

Then, the structure of $\delta_{\mathcal{P}}$ -prolongation sequence in Example 2.10 induces a structure of $\delta_{\mathcal{P}}$ -prolongation sequence on the sequence of rings (A^r) . Indeed, the p_k -derivation δ_{p_k} sends the ideal

$$I_r := (\delta_{\mathcal{P}}^i T : i \leq r) \subset A^r$$

into the ideal $I_{r+e_k} \subset A^{r+e_k}$; cf. (13). \square

2.2. $\delta_{\mathcal{P}}$ -jet spaces. As in the case of a single prime [5], we now have the following existence result for a universal prolongation sequence.

Proposition 2.12. *Let A^0 be a finitely generated A_0 -algebra. Then there exists a $\delta_{\mathcal{P}}$ -prolongation sequence A^* over A_0 , with A^r finitely generated over A_0 , satisfying the following property: for any $\delta_{\mathcal{P}}$ -prolongation system B^* over A_0 and any A_0 -algebra homomorphism $u : A^0 \rightarrow B^0$, there exists a unique morphism of $\delta_{\mathcal{P}}$ -prolongation systems $u^* : A^* \rightarrow B^*$ such that $u^0 = u$.*

Proof. We express the finitely generated algebra A^0 as

$$A^0 = \frac{A_0[x]}{(f)}$$

for a tuple of indeterminates x , and a tuple of polynomials f . Then we set

$$A^r = \frac{A_0[\delta_{\mathcal{P}}^i x : i \leq r]}{(\delta_{\mathcal{P}}^i f : i \leq r)}.$$

Using Example 2.10 and the identities (14), we check easily that $A^* = (A^r)$ has the universality property in the statement. \square

Definition 2.13. Let X be an affine scheme of finite type over A_0 . Let $A^0 = \mathcal{O}(X)$ and let A^r be as in Proposition 2.12. Then the scheme

$$\mathcal{J}_{\mathcal{P}}^r(X) := \operatorname{Spec} A^r$$

is called the $\delta_{\mathcal{P}}$ -jet space of order r of X . \square

By the universality property in Proposition 2.12, up to isomorphism the scheme $\mathcal{J}_{\mathcal{P}}^r(X)$ depends on X alone, and is functorial in X : for any morphism $X \rightarrow Y$ of affine schemes of finite type, there are induced morphisms of schemes

$$\mathcal{J}_{\mathcal{P}}^r(Y) \rightarrow \mathcal{J}_{\mathcal{P}}^r(X).$$

Remark 2.14. In the case when \mathcal{P} consists of a single prime p , the p -adic completions of the schemes $\mathcal{J}_p^r(X)$ were introduced and thoroughly studied in [5, 6, 9]. For arbitrary \mathcal{P} , the schemes $\mathcal{J}_{\mathcal{P}}^r(X)$ above were independently introduced by Borger in [2], where they are denoted by $W_{r*}(X)$.

Lemma 2.15. *Let X be an affine scheme of finite type over A_0 and let $Y \subset X$ be a principal open set of X , $\mathcal{O}(Y) = \mathcal{O}(X)_f$. Then $\mathcal{O}(\mathcal{J}_{\mathcal{P}}^r(Y)) \simeq \mathcal{O}(\mathcal{J}_{\mathcal{P}}^r(X))_{f_r}$ where $f_r = \prod_{i \leq r} \phi_{\mathcal{P}}^i(f)$. In particular, the induced morphism $\mathcal{J}_{\mathcal{P}}^r(Y) \rightarrow \mathcal{J}_{\mathcal{P}}^r(X)$ is an open immersion whose image is principal, and if we view this morphism as an inclusion and $Z \subset X$ is another principal open set, then we have that*

$$\mathcal{J}_{\mathcal{P}}^r(Y \cap Z) = \mathcal{J}_{\mathcal{P}}^r(Y) \cap \mathcal{J}_{\mathcal{P}}^r(Z).$$

Proof. We can check easily that $\mathcal{O}(\mathcal{J}_{\mathcal{P}}^r(X))_{f_r}$ has the universality property of $\mathcal{O}(\mathcal{J}_{\mathcal{P}}^r(Y))$. The δ_{p_k} -derivations on $\mathcal{O}(\mathcal{J}_{\mathcal{P}}^r(X))_{f_r}$ are defined via the formula

$$\delta_{p_k} \left(\frac{a}{b} \right) = \frac{b^{p_k} \delta_{p_k} a - a^{p_k} \delta_{p_k} b}{b^p \phi_{p_k}(b)}.$$

\square

Definition 2.16. Let X be a scheme of finite type over A_0 . An affine open covering

$$(16) \quad X = \bigcup_{i=1}^m X_i$$

is called *principal* if

$$(17) \quad X_i \cap X_j \text{ is principal in both } X_i \text{ and } X_j,$$

for all $i, j = 1, \dots, m$.

Let $c = \{X_i\}_{i=1}^m$ be a principal covering of X . We define the $\delta_{\mathcal{P}}$ -jet space of order r of X with respect to the covering c by gluing $\mathcal{J}_{\mathcal{P}}^r(X_i)$ along $\mathcal{J}_{\mathcal{P}}^r(X_i \cap X_j)$, and denote this jet space by $\mathcal{J}_{c, \mathcal{P}}^r(X)$. \square

In the case when X is affine, we may use the *tautological covering* c of X , that is to say, the covering of X with a single open set, the set X itself. Then the space $\mathcal{J}_{c, \mathcal{P}}^r(X)$ coincides with the space $\mathcal{J}_{\mathcal{P}}^r(X)$ of Definition 2.13.

Notice that any quasi-projective scheme X admits a principal covering.

Remark 2.17. Generally speaking, the scheme $\mathcal{J}_{c, \mathcal{P}}^r(X)$ in Definition 2.16 depends on the covering c . For instance, if $X = \operatorname{Spec} A_0[x]$ is the affine line, we may consider the principal cover \tilde{c} consisting of the two open sets $X_0 = \operatorname{Spec} A_0[x, 1/x]$

and $X_1 = \text{Spec } A_0[x, 1/(x-1)]$, respectively. Then, if $\mathcal{P} = \{p\}$, the scheme $\mathcal{J}_{c,\mathcal{P}}^1(X)$ is the union

$$(18) \quad \text{Spec } A_0 \left[x, \delta_p x, \frac{1}{x(x^p + p\delta_p x)} \right] \cup \text{Spec } A_0 \left[x, \delta_p x, \frac{1}{(x-1)(x^p + p\delta_p x - 1)} \right]$$

inside the plane

$$(19) \quad \text{Spec } A_0[x, \delta_p x],$$

while the scheme $\mathcal{J}_{\mathcal{P}}^1(X)$ corresponding to the tautological covering c of X is the whole plane (19). But the union (18) does not equal this plane. For the K_0 -point of (19) with coordinates $x = 0$, $\delta_p x = 1/p$ does not belong to (18).

This phenomenon is similar to problems encountered in [12, 16, 2] and, as in loc.cit., there are ways to fix this anomaly at the cost of developing more technology. For our purposes here, these covering dependent rudimentary definition of jet spaces will be sufficient. Indeed some of the basic rings of functions that we are going to consider will be independent of the coverings; cf. Proposition 2.22. However, let us observe that the endofunctor $X \mapsto \mathcal{J}_{\mathcal{P}}^r(X)$ on the category of affine schemes of finite type over A_0 that we consider here has been extended in [2] to an endofunctor on the category of algebraic spaces over A_0 . \square

Remark 2.18. If X is a closed subscheme of a projective space over A_0 , then X has a natural principal covering c where the open sets are the complements of the coordinate hyperplanes. We may thus attach to X the scheme $\mathcal{J}_{c,\mathcal{P}}^r(X)$ corresponding to this cover. In general, such a definition depends upon the embedding of X into a projective space. \square

Remark 2.19. When $\mathcal{P} = \{p\}$ consists of a single prime p , then the p -adic completion $\mathcal{J}_{c,\mathcal{P}}^r(X)^{\widehat{}}$ (cf. the notation in §1.3) coincides with the p -jet space $J_p^r(X)$ over \mathbb{Z}_p in [5, 9], and therefore, it depends on X only and not on the covering c . For $\mathcal{P} = \{p_1, \dots, p_d\}$, we obtain natural morphisms

$$\mathcal{J}_{c,\mathcal{P}}^{je_k}(X)^{\widehat{}} \simeq \mathcal{J}_{c,\{p_k\}}^j(X)$$

for each $j \in \mathbb{Z}_+$, and consequently, natural morphisms

$$(20) \quad \mathcal{J}_{c,\mathcal{P}}^r(X)^{\widehat{}} \rightarrow \mathcal{J}_{c,\mathcal{P}}^{r_k e_k}(X)^{\widehat{}} = \mathcal{J}_{c,\{p_k\}}^{r_k}(X)^{\widehat{}} = J_{p_k}^{r_k}(X).$$

\square

Remark 2.20. Let X be an affine scheme with its tautological cover c . Then the system $(\mathcal{O}(\mathcal{J}^r(X))^{\widehat{}})$ has a natural structure of $\delta_{\mathcal{P}}$ -prolongation system. It has a universality property analogue to that in Proposition 2.12 with respect to $\delta_{\mathcal{P}}$ -prolongation systems B^* of p_k -adically complete rings. \square

We have the following structure result for p_k -adic completions of $\delta_{\mathcal{P}}$ -jet spaces.

Lemma 2.21. *Let us assume that $X = \text{Spec } A^0$ is affine over A_0 , and let $A_0[T] \rightarrow A^0$ be an étale map, where T is a tuple of variables. Consider X with its tautological cover. Then there is a natural isomorphism*

$$(21) \quad \mathcal{O}(\mathcal{J}_{\mathcal{P}}^r(X))^{\widehat{}} \simeq (\otimes_{i \leq r - r_k e_k} A_i^0) [\delta_{p_k}^j \phi_{p_k}^i T : j \geq 1, i + je_k \leq r]^{\widehat{}},$$

where $A_i^0 = A^0$, and $\otimes = \otimes_{A_0}$.

Proof. If $A^0 = A_0[x]/(f)$, we set $A_i^0 := A_0[\phi_{\mathcal{P}}^i x]/(\phi_{\mathcal{P}}^i f)$. Then we can check that the right hand side of (21) has the universality property of the left hand side, as explained in Remark 2.20. For this, we need to use the fact that p_l is invertible in any p_k -adic completion, $l \neq k$, and use the argument in Proposition 3.13 in [9]. \square

Proposition 2.22. *Let us assume that X is a smooth scheme of finite type, quasi-projective and with connected geometric fibers over $A_0 = \mathbb{Z}_{(\mathcal{P})}$. Let c and c' be two principal coverings of X , and $\mathcal{J}_{c,\mathcal{P}}^r(X)$ and $\mathcal{J}_{c',\mathcal{P}}^r(X)$ be the corresponding jet spaces. Then, there is a natural isomorphism*

$$\mathcal{O}(\mathcal{J}_{c,\mathcal{P}}^r(X)^{\widehat{p_k}}) \simeq \mathcal{O}(\mathcal{J}_{c',\mathcal{P}}^r(X)^{\widehat{p_k}}).$$

Thus, we may drop the covering from the notation and denote the isomorphism class of rings $\mathcal{O}(\mathcal{J}_{c,\mathcal{P}}^r(X)^{\widehat{p_k}})$ simply by $\mathcal{O}(\mathcal{J}_{\mathcal{P}}^r(X)^{\widehat{p_k}})$.

Proof. Let us assume that $X = \text{Spec } A^0$ has an étale map to an affine space over A_0 , and consider a covering $X = \cup X_j$ where $X_j = \text{Spec } A_{f_j}^0$. We let $Y = \mathcal{J}_{\mathcal{P}}^r(X)$ and $Y_j = \mathcal{J}_{\mathcal{P}}^r(X_j)$ be the jet spaces corresponding to the tautological coverings of X and X_j , respectively, and set $U = \cup Y_j$. We claim that the restriction map $\mathcal{O}(Y^{\widehat{p_k}}) \rightarrow \mathcal{O}(U^{\widehat{p_k}})$ is an isomorphism, fact that is easily seen to imply the statement of the Proposition.

For the proof of this claim, it is enough to check that the map $\mathcal{O}(\overline{Y}) \rightarrow \mathcal{O}(\overline{U})$ is an isomorphism, where $\overline{Y} := Y \otimes_{\mathbb{F}_p} \mathbb{F}_{p_k}$ and $\overline{U} := U \otimes_{\mathbb{F}_p} \mathbb{F}_{p_k}$, respectively. Observe that \overline{Y} is smooth over \mathbb{F}_p (cf. Lemma 2.21), and that \overline{U} is an open subset of \overline{Y} (cf. Lemma 2.15). Thus, it suffices to check that $\overline{Y} \setminus \overline{U}$ has codimension ≥ 2 in \overline{Y} . By Lemmas 2.21 and 2.15, we have identifications $\overline{Y} = \overline{X}^a \times \mathbb{A}^b$ and $\overline{Y}_j = \overline{X}_j^a \times \mathbb{A}^b$, where \mathbb{A}^b is the affine space of dimension b over \mathbb{F}_{p_k} , and \overline{X}^a and \overline{X}_j^a are the a -fold products of \overline{X} and X_j , respectively. Let us set $H_j = \overline{X} \setminus \overline{X}_j$. We may assume that all the H_j s are non-empty and different from \overline{X} . So each H_j is a hypersurface in \overline{X} . We have that

$$\overline{X}^a \setminus (\cup \overline{X}_j^a) \subset \bigcup_{1 \leq i_1, \dots, i_a \leq a} S_{i_1, \dots, i_a}$$

where

$$S_{i_1, \dots, i_a} = \{(P_1, \dots, P_a) \in \overline{X}^a : P_{i_k} \in H_k, 1 \leq k \leq a\}.$$

Now, if $i_1 = \dots = i_a$ then $S_{i_1, \dots, i_a} = \emptyset$. On the other hand, if at least two of the indices i_1, \dots, i_a are different, then S_{i_1, \dots, i_a} has codimension ≥ 2 in \overline{X}^a . This implies that $\overline{Y} \setminus \overline{U}$ has codimension ≥ 2 in \overline{Y} , as desired. \square

2.3. $\delta_{\mathcal{P}}$ -functions. We would like to see now how we can “glue” elements in various completions of a given ring. We achieve this through the following definitions.

Definition 2.23. Let A be a ring, I be an ideal in A , and $\mathcal{P} = \{p_1, \dots, p_d\}$ be a finite set of primes in \mathbb{Z} that are non-invertible in A/I . We say that a family

$$(22) \quad f = (f_k) \in \prod_{k=1}^d A^{\widehat{p_k}}$$

can be *analytically continued along I* if there exists $f_0 \in A^{\widehat{I}}$ such that the images of f_0 and f_k in the ring $A^{\widehat{(p_k, I)}}$ coincide for each $k = 1, \dots, d$. In that case, we say

that f_0 represents f . If $X = \operatorname{Spec} A$, we denote by $\mathcal{O}_{I,\mathcal{P}}(X)$ the ring of families (22) that can be analytically continued along I . \square

From this point on, the $\delta_{\mathcal{P}}$ -ring A_0 under consideration will be just $\mathbb{Z}_{(\mathcal{P})}$. Cf. the notation in §1.3.

Given a $\mathbb{Z}_{(\mathcal{P})}$ -point $P : \operatorname{Spec} \mathbb{Z}_{(\mathcal{P})} \rightarrow X$, by the universality property, we obtain a unique lift to a point $P^r : \operatorname{Spec} \mathbb{Z}_{(\mathcal{P})} \rightarrow \mathcal{J}_{\mathcal{P}}^r(X)$ that is compatibly with the action of $\delta_{\mathcal{P}}$.

Definition 2.24. Let X be an affine scheme of finite type over $\mathbb{Z}_{(\mathcal{P})}$, P be a $\mathbb{Z}_{(\mathcal{P})}$ -point $P : \operatorname{Spec} \mathbb{Z}_{(\mathcal{P})} \rightarrow X$, and $P^r : \operatorname{Spec} \mathbb{Z}_{(\mathcal{P})} \rightarrow \mathcal{J}_{\mathcal{P}}^r(X)$ be its unique lift compatible with the action of $\delta_{\mathcal{P}}$. By abuse of notation, we also denote by $P^r \subset \mathcal{O}(\mathcal{J}_{\mathcal{P}}^r(X))$ the ideal of the image of P^r . By a $\delta_{\mathcal{P}}$ -function on X that is analytically continued along P we mean a family

$$(23) \quad f = (f_k) \in \prod_{k=1}^d \mathcal{O}(\mathcal{J}_{\mathcal{P}}^r(X))^{\widehat{P^k}}$$

that can be analytically continued along P^r . We denote by $\mathcal{O}_{P,\mathcal{P}}^r(X)$ the ring of all $\delta_{\mathcal{P}}$ -functions on X that are analytically continued along P . \square

Let us recall that by Definition 2.23, if $f = (f_k) \in \prod_{k=1}^d \mathcal{O}(\mathcal{J}_{\mathcal{P}}^r(X))^{\widehat{P^k}}$ is analytically continued along P there exists an element

$$f_0 \in \mathcal{O}(\mathcal{J}_{\mathcal{P}}^r(X))^{\widehat{P^r}}$$

that represents f such that the images of f_0 and f_k in

$$\mathcal{O}(\mathcal{J}_{\mathcal{P}}^r(X))^{\widehat{(P^k, P^r)}}$$

coincide for each $k = 1, \dots, d$. Thus, the the ring of all $\delta_{\mathcal{P}}$ -functions on X that are analytically continued along P is

$$\mathcal{O}_{P,\mathcal{P}}^r(X) := \mathcal{O}_{P^r,\mathcal{P}}(\mathcal{J}_{\mathcal{P}}^r(X)).$$

In the next section, we shall encounter natural examples of families (23) that can be analytically continued along a point P but not along others, as well as examples of such families that can be analytically continued along *any* point.

Under very general hypotheses on X and P that suffice for our applications, this analytical continuation concept admits a practical description that we now explain.

Definition 2.25. Let X be a smooth affine scheme over $\mathbb{Z}_{(\mathcal{P})}$. A $\mathbb{Z}_{(\mathcal{P})}$ -point $P : \operatorname{Spec} \mathbb{Z}_{(\mathcal{P})} \rightarrow X$ of X is called *uniform* if there exists an étale map of $\mathbb{Z}_{(\mathcal{P})}$ -algebras $\mathbb{Z}_{(\mathcal{P})}[T] \rightarrow \mathcal{O}(X)$, where T is a tuple of indeterminates, such that the ideal of the image of P in $\mathcal{O}(X)$ is generated by T . We refer to T as *uniform coordinates*. Then T is a regular sequence in $\mathcal{O}(X)$, and the graded ring associated to the ideal (T) in $\mathcal{O}(X)$ is isomorphic to $\mathbb{Z}_{(\mathcal{P})}[T]$; cf. [22], p. 125. We have that

$$\mathcal{O}(X)^{\widehat{T}} \simeq \mathbb{Z}_{(\mathcal{P})}[[T]],$$

and similarly that

$$\mathcal{O}(X)^{\widehat{(P^k, T)}} \simeq \mathbb{Z}_{P^k}[[T]].$$

For a general scheme X , we say that a $\mathbb{Z}_{(\mathcal{P})}$ -point P of X is *uniform* if there exists an affine open set $X_1 \subset X$ that contains P such that P is uniform in X_1 . \square

Uniformity is a property that holds “generically” in the following sense: if $X_{\mathbb{Q}}$ is a smooth scheme over \mathbb{Q} and $P_{\mathbb{Q}}$ is a \mathbb{Q} -point of $X_{\mathbb{Q}}$, then there exists an integer

$N \geq 1$ and models $X_{\mathbb{Z}[1/N]}$, $P_{\mathbb{Z}[1/N]}$ over $\mathbb{Z}[1/N]$ such that, for any set of primes \mathcal{P} that are coprime to N , the induced point $P_{\mathbb{Z}(\mathcal{P})}$ of $X_{\mathbb{Z}(\mathcal{P})}$ is uniform.

Remark 2.26. Let X be a smooth affine scheme with connected geometric fibers over $\mathbb{Z}(\mathcal{P})$ provided with its tautological cover, and let P be a uniform point with uniform coordinates T . We consider the corresponding étale map $\mathbb{Z}(\mathcal{P})[T] \rightarrow \mathcal{O}(X)$, and let t be the tuple of indeterminates $(\delta_{\mathcal{P}}^i T)_{i \leq r}$. By [2], Proposition 8.2, the map

$$\mathbb{Z}(\mathcal{P})[t] \rightarrow \mathcal{O}(\mathcal{J}_{\mathcal{P}}^r(X))$$

is étale, and in particular, the r -jet space $\mathcal{J}_{\mathcal{P}}^r(X)$ is smooth over $\mathbb{Z}(\mathcal{P})$. Once again, we let P^r be the canonical lift of P . Then P^r is uniform with uniform coordinates t . By Definition 2.25, we have that

$$\begin{aligned} \mathcal{O}(\mathcal{J}^r(X))^{\widehat{t}} &\simeq \mathbb{Z}(\mathcal{P})[[t]], \\ \mathcal{O}(\mathcal{J}^r(X))^{\widehat{(P_k, t)}} &\simeq \mathbb{Z}_{p_k}[[t]]. \end{aligned}$$

Hence, by Definition 2.24, a family

$$(24) \quad f = (f_k) \in \prod_{k=1}^d \mathcal{O}(\mathcal{J}_{\mathcal{P}}^r(X))^{\widehat{P_k}}$$

is a $\delta_{\mathcal{P}}$ -function on X that is analytically continued along P if there exists $f_0 \in \mathbb{Z}(\mathcal{P})[[t]]$ such that the images of f_k and f_0 in $\mathbb{Z}_{p_k}[[t]]$ coincide for each $k = 1, \dots, d$. Notice that such an f_0 is uniquely determined by f . \square

Definition 2.27. Let X be a smooth quasi-projective scheme over $\mathbb{Z}(\mathcal{P})$ with geometrically connected fibers, and let P be a uniform point in some affine open set X_1 . Let us denote by $\mathcal{O}_{P, \mathcal{P}}^r(X)$ the preimage of $\mathcal{O}_{P, \mathcal{P}}^r(X_1)$ via the restriction map

$$\prod_{k=1}^d \mathcal{O}(\mathcal{J}_{\mathcal{P}}^r(X))^{\widehat{P_k}} \rightarrow \prod_{k=1}^d \mathcal{O}(\mathcal{J}_{\mathcal{P}}^r(X_1))^{\widehat{P_k}} = \prod_{k=1}^d \mathcal{O}(\mathcal{J}_{\mathcal{P}}^r(X_1))^{\widehat{P_k}}.$$

Elements of this ring $\mathcal{O}_{P, \mathcal{P}}^r(X)$ are referred to as $\delta_{\mathcal{P}}$ -functions of order r on X that are analytically continued along P . We define the ring of $\delta_{\mathcal{P}}$ -functions on X that are analytically continued along P by

$$\mathcal{O}_{P, \mathcal{P}}^{\infty}(X) := \varinjlim \mathcal{O}_{P, \mathcal{P}}^r(X),$$

with its natural $\delta_{\mathcal{P}}$ -ring structure. \square

Example 2.28. Let $X = \mathbb{A}^n = \text{Spec } \mathbb{Z}(\mathcal{P})[T]$ be the affine space of dimension n over $\mathbb{Z}(\mathcal{P})$, where T is an n -tuple of indeterminates. We let P be the zero section, and let t be the tuple of indeterminates $(\delta_{\mathcal{P}}^i T)_{i \leq r}$. If v_{p_k} denotes the p_k -adic valuation, then

$$\mathcal{O}_{P, \mathcal{P}}^r(\mathbb{A}^n) \simeq \left\{ \sum a_j t^j \in \mathbb{Z}(\mathcal{P})[[t]] : \lim_{|j| \rightarrow \infty} v_{p_k}(a_j) \rightarrow \infty \text{ for each } k \right\}.$$

\square

Remark 2.29. Let $u : X \rightarrow Y$ be a morphism of quasi-projective schemes over $\mathbb{Z}(\mathcal{P})$. Assume that X and Y are smooth with connected geometric fibers. Let P be a uniform point of X such that $u(P)$ is a uniform point of Y . Then there is a naturally induced homomorphism

$$u^* : \mathcal{O}_{u(P), \mathcal{P}}^r(Y) \rightarrow \mathcal{O}_{P, \mathcal{P}}^r(X).$$

□

Remark 2.30. By Lemma 2.21 it follows that if X is a smooth quasi-projective scheme with connected geometric fibers over $\mathbb{Z}_{(\mathcal{P})}$, and P is a uniform point of X , then the induced homomorphisms

$$\mathcal{O}(\mathcal{J}_{\mathcal{P}}^r(X)^{\widehat{p_k}}) \rightarrow \mathbb{Z}_{p_k}[[\delta_{\mathcal{P}}^i T : i \leq r]]$$

are injective. In particular, the homomorphism

$$\mathcal{O}_{P,\mathcal{P}}^r(X) \rightarrow \mathbb{Z}_{(\mathcal{P})}[[\delta_{\mathcal{P}}^i T : i \leq r]]$$

is injective. □

Definition 2.31. Let X be a smooth quasi-projective scheme over $\mathbb{Z}_{(\mathcal{P})}$ with geometrically connected fibers, and let P be a uniform point of X . We consider a $\delta_{\mathcal{P}}$ -function of order r analytically continued along P , $f = (f_k) \in \mathcal{O}_{P,\mathcal{P}}^r(X)$. If A is a $\delta_{\mathcal{P}}$ -ring over $\mathbb{Z}_{(\mathcal{P})}$, then there is an induced map of sets

$$f_A : X(A_{\mathcal{P}}) \rightarrow A_{\mathcal{P}},$$

where $A_{\mathcal{P}} = \prod A^{\widehat{p_k}}$ (cf. the notation in §1.3), defined as follows. Let $Q = (Q_k) \in X(A_{\mathcal{P}}) = \prod_{k=1}^d X(A^{\widehat{p_k}})$ be a point. Since each $A^{\widehat{p_k}}$ is a $\delta_{\mathcal{P}}$ -ring (cf. Remark 2.4), by the universality property of $\delta_{\mathcal{P}}$ -jet spaces the morphism $Q_k : \text{Spec } A^{\widehat{p_k}} \rightarrow X$ lifts to a morphism $Q_k^r : \text{Spec } A^{\widehat{p_k}} \rightarrow \mathcal{J}_{c,\mathcal{P}}^r(X)$, where c is any principal covering. Since $A^{\widehat{p_k}}$ is p_k -adically complete, we obtain an induced morphism $\widehat{Q}_k^r : \text{Spf } A^{\widehat{p_k}} \rightarrow \mathcal{J}_{c,\mathcal{P}}^r(X)^{\widehat{p_k}}$, and therefore a homomorphism $\mathcal{O}(\mathcal{J}_{\mathcal{P}}^r(X)^{\widehat{p_k}}) \rightarrow A^{\widehat{p_k}}$. We denote the image of f_k under this last homomorphism by $f_k(Q_k)$. Then

$$f_A(Q) := (f_k(Q_k)) \in A_{\mathcal{P}}.$$

□

Using the map f_A of Definition 2.31 above, we may speak of the *space of solutions*, $f_A^{-1}(0)$.

Notice that the definition of f_A is functorial in A . For if $A \rightarrow B$ is a morphism of $\delta_{\mathcal{P}}$ -rings, then the corresponding maps f_A and f_B are compatible with each other.

2.4. $\delta_{\mathcal{P}}$ -characters. Let G be a smooth group scheme over $\mathbb{Z}_{(\mathcal{P})}$ with multiplication $\mu : G \times G \rightarrow G$. Here, $\times = \times_{\mathbb{Z}_{(\mathcal{P})}}$. If G is affine, by the universality property we have that $\mathcal{J}_{\mathcal{P}}^r(G)$ is a group scheme over $\mathbb{Z}_{(\mathcal{P})}$. In the non-affine case this is not a priori true because our definitions are covering dependent. However, we attach to this general situation a formal group law as follows.

Let Z be the identity $\text{Spec } \mathbb{Z}_{(\mathcal{P})} \rightarrow G$, and assume Z is uniform, with uniform coordinates T . Then $Z \times Z \subset G \times G$ is a uniform point with uniform coordinates T_1, T_2 induced by T . We have an induced homomorphism $\mathbb{Z}_{(\mathcal{P})}[[T]] \rightarrow \mathbb{Z}_{(\mathcal{P})}[[T_1, T_2]]$ that sends the ideal (T) into the ideal (T_1, T_2) . By the universality property applied to the restriction $\mathbb{Z}_{(\mathcal{P})}[T] \rightarrow \mathbb{Z}_{(\mathcal{P})}[[T_1, T_2]]$, we obtain morphisms

$$\mathbb{Z}_{(\mathcal{P})}[\delta_{\mathcal{P}}^i T : i \leq r] \rightarrow \mathbb{Z}_{(\mathcal{P})}[[\delta_{\mathcal{P}}^i T_1, \delta_{\mathcal{P}}^i T_2 : i \leq r]]$$

that send the ideal generated by the variables into the corresponding ideal generated by the variables. Thus, we have an induced morphism

$$\mathbb{Z}_{(\mathcal{P})}[[\delta_{\mathcal{P}}^i T : i \leq r]] \rightarrow \mathbb{Z}_{(\mathcal{P})}[[\delta_{\mathcal{P}}^i T_1, \delta_{\mathcal{P}}^i T_2 : i \leq r]].$$

We call \mathcal{G}^r the image of the variables $\{\delta_{\mathcal{P}}^i T : i \leq r\}$ under this last homomorphism. Then the tuple \mathcal{G}^r is a formal group law over $\mathbb{Z}_{(\mathcal{P})}$.

Definition 2.32. Let G be a quasi-projective smooth group scheme over $\mathbb{Z}_{(\mathcal{P})}$ with geometrically connected fibers. Let us assume that the identity is a uniform point. Then there are homomorphisms

$$\mu^*, pr_1^*, pr_2^* : \mathcal{O}_{Z, \mathcal{P}}^r(G) \rightarrow \mathcal{O}_{Z \times Z, \mathcal{P}}^r(G \times G)$$

induced by the product μ and the two projections. We say that a $\delta_{\mathcal{P}}$ -function $f \in \mathcal{O}_{Z, \mathcal{P}}^r(G)$ of order r on G that is analytically continued along Z is a $\delta_{\mathcal{P}}$ -character of order r on G if

$$\mu^* f = pr_1^* f + pr_2^* f.$$

We denote by $\mathbf{X}_{\mathcal{P}}^r(G)$ the group of $\delta_{\mathcal{P}}$ -characters of order r on G . We define the group of $\delta_{\mathcal{P}}$ -characters on G to be

$$\mathbf{X}_{\mathcal{P}}^{\infty}(G) := \varinjlim \mathbf{X}_{\mathcal{P}}^r(G).$$

□

By Remark 2.30, the condition that f be a $\delta_{\mathcal{P}}$ -character of order r on G is that there exists

$$f_0 \in \mathbb{Z}_{(\mathcal{P})}[[\delta_{\mathcal{P}}^i T : i \leq r]]$$

that represents f such that

$$(25) \quad f_0(\mathcal{G}^r(T_1, T_2)) = f_0(T_1) + f_0(T_2).$$

Here, \mathcal{G}^r is the corresponding formal group law, and $f_0(T)$ stands for $f_0(\dots, \delta_{\mathcal{P}}^i T, \dots)$.

The group $\mathbf{X}_{\mathcal{P}}^r(G)$ of $\delta_{\mathcal{P}}$ -characters of order r on G is a subgroup of the additive group of the ring $\mathcal{O}_{Z, \mathcal{P}}^r(G)$. The group $\mathbf{X}_{\mathcal{P}}^{\infty}(G)$ of $\delta_{\mathcal{P}}$ -characters on G is a subgroup of $\mathcal{O}_{Z, G}^{\infty}(G)$.

For any $\delta_{\mathcal{P}}$ -character $\psi \in \mathbf{X}_{\mathcal{P}}^r(G)$ and any $\delta_{\mathcal{P}}$ -ring A , there is an induced group homomorphism

$$(26) \quad \psi_A : G(A_{\mathcal{P}}) \rightarrow A_{\mathcal{P}},$$

where $A_{\mathcal{P}}$ is viewed as a group with respect to addition. We may therefore speak of the *group of solutions* $\text{Ker } \psi_A$.

Once again, the mapping $A \mapsto \psi_A$ is functorial in A .

Definition 2.33. Let P be a $\mathbb{Z}_{(\mathcal{P})}$ -point of G . We say that a $\delta_{\mathcal{P}}$ -character $\psi \in \mathbf{X}_{\mathcal{P}}^r(G)$ can be *analytically continued along* P if $\psi \in \mathcal{O}_{P, \mathcal{P}}^r(G)$. □

By the mere definition, any $\delta_{\mathcal{P}}$ -character can be analytically continued along the identity Z . Later on we will address the question of when a $\delta_{\mathcal{P}}$ -character can be analytically continued along points other than Z .

3. MAIN RESULTS

In this section we determine all the $\delta_{\mathcal{P}}$ -characters on the additive group \mathbb{G}_a , the multiplicative group \mathbb{G}_m , and elliptic curves E over $\mathbb{Z}_{(\mathcal{P})}$.

3.1. The additive group. We consider the additive group scheme over $\mathbb{Z}_{(\mathcal{P})}$,

$$\mathbb{G}_a := \text{Spec } \mathbb{Z}_{(\mathcal{P})}[x].$$

The zero section is uniform, with uniform coordinate $T = x$. We have

$$\mathcal{O}(\mathcal{J}_{\mathcal{P}}^r(\mathbb{G}_a)) = \mathbb{Z}_{(\mathcal{P})}[\delta_{\mathcal{P}}^i x : i \leq r].$$

Let us consider the polynomial ring

$$\mathbb{Z}_{(\mathcal{P})}[\phi_{\mathcal{P}}] := \mathbb{Z}_{(\mathcal{P})}[\phi_{p_l} : l \in I] = \sum_{n \in \mathcal{N}} \mathbb{Z}_{(\mathcal{P})} \phi_n,$$

where $I := \{1, \dots, d\}$, \mathcal{N} is the monoid of the natural numbers generated by \mathcal{P} , and the ϕ_{p_l} s are commuting variables. For $i = (i_1, \dots, i_d) \in \mathbb{Z}_+^d$ we set $\mathcal{P}^i = p_1^{i_1} \dots p_d^{i_d}$. If $n = \mathcal{P}^i$ we set $\phi_n = \phi_{\mathcal{P}}^i = \phi_{p_1}^{i_1} \dots \phi_{p_d}^{i_d}$. We will view the ring $\mathbb{Z}_{(\mathcal{P})}[\phi_{\mathcal{P}}]$ as the *ring of symbols*; cf. [10]. Let $r \in \mathbb{Z}_+^d$ and

$$\bar{\psi} := \sum_{n | \mathcal{P}^r} c_n \phi_n \in \mathbb{Z}_{(\mathcal{P})}[\phi_{\mathcal{P}}].$$

We may consider the element $\psi = \bar{\psi}x \in \mathcal{O}(\mathcal{J}^r(\mathbb{G}_a))$ and identify it with an element

$$\psi \in \prod_{k=1}^d \mathcal{O}(\mathcal{J}_{\mathcal{P}}^r(\mathbb{G}_a))^{\widehat{\mathcal{P}^k}}.$$

via the diagonal embedding. Then $\bar{\psi}$ clearly defines a $\delta_{\mathcal{P}}$ -character on \mathbb{G}_a . We will usually identify ψ and $\bar{\psi}$.

As we see now, the following result is easy to prove for the additive group. Later on, we shall prove a less elementary analogue for the multiplicative group \mathbb{G}_m , and for elliptic curves.

Theorem 3.1. *Let ψ be a $\delta_{\mathcal{P}}$ -character on \mathbb{G}_a of order r . Then*

- (1) *ψ can be uniquely written as $\psi = \bar{\psi}x$ with*

$$\bar{\psi} = \left(\sum_{n | \mathcal{P}^r} c_n \phi_n \right) x, \quad c_n \in \mathbb{Z}_{(\mathcal{P})}.$$

- (2) *ψ can be analytically continued along any $\mathbb{Z}_{(\mathcal{P})}$ -point P of \mathbb{G}_a .*

Corollary 3.2. *The group of $\delta_{\mathcal{P}}$ -characters $\mathbf{X}_{\mathcal{P}}^{\infty}(\mathbb{G}_a)$ is a free $\mathbb{Z}_{(\mathcal{P})}[\phi_{\mathcal{P}}]$ -module of rank one with basis x .*

Proof of Theorem 3.1. Let $\psi \in \mathbf{X}_{\mathcal{P}}^r(\mathbb{G}_a)$ be represented by a series $\psi_0 \in \mathbb{Z}_{(\mathcal{P})}[[\delta_{\mathcal{P}}^i x : i \leq r]]$. We view ψ_0 as an element of

$$\mathbb{Q}[[\delta_{\mathcal{P}}^i x : i \leq r]] = \mathbb{Q}[[\phi_{\mathcal{P}}^i x : i \leq r]].$$

We have that

$$\begin{aligned} \psi_0(\dots, \phi_{\mathcal{P}}^i x_1 + \phi_{\mathcal{P}}^i x_2, \dots) &= \psi_0(\dots, \phi_{\mathcal{P}}^i (x_1 + x_2), \dots) \\ &= \psi_0(\dots, \phi_{\mathcal{P}}^i x_1, \dots) + \psi_0(\dots, \phi_{\mathcal{P}}^i x_2, \dots). \end{aligned}$$

Then it follows that $\psi_0 = \sum c_n \phi_n x$ with $c_n \in \mathbb{Q}$.

Now notice that

$$\phi_n x \equiv x^n \pmod{(\delta_{\mathcal{P}}^i x : i \leq r)}$$

in the ring $\mathbb{Q}[[\delta_{\mathcal{P}}^i x : i \leq r]]$. We get that

$$(\psi_0)_{|\delta_{\mathcal{P}}^i x=0; 0 \leq i \leq r} = \sum c_n x^n.$$

It follows that $c_n \in \mathbb{Z}_{(\mathcal{P})}$, and the first assertion is proved.

The second assertion is obvious. □

3.2. The multiplicative group. We now consider the multiplicative group scheme over $\mathbb{Z}_{(\mathcal{P})}$,

$$\mathbb{G}_m := \operatorname{Spec} \mathbb{Z}_{(\mathcal{P})}[x, x^{-1}].$$

The zero section is uniform, with uniform coordinate $T = x - 1$.

We recall that the formal group law \mathcal{G}^0 corresponding to T is

$$\mathcal{G}^0(T_1, T_2) = T_1 + T_2 + T_1 T_2,$$

and the logarithm of this formal group law is given by the series

$$l_{\mathbb{G}_m}(T) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{T^n}{n} \in \mathbb{Q}[[T]].$$

By Lemma 2.15, we have that

$$\mathcal{O}(\mathcal{J}_{\mathcal{P}}^r(\mathbb{G}_m)) = \mathbb{Z}_{(\mathcal{P})} \left[\delta_{\mathcal{P}}^i x, \frac{1}{\phi_{\mathcal{P}}^i(x)} : i \leq r \right].$$

For each k , we consider the series

$$\psi_{p_k} = \psi_{p_k}^1 := \sum_{n=1}^{\infty} (-1)^{n-1} \frac{p_k^{n-1}}{n} \left(\frac{\delta_{p_k} x}{x^{p_k}} \right)^n \in \mathcal{O}(\mathcal{J}_{\mathcal{P}}^{e_k}(\mathbb{G}_m))^{\widehat{p_k}},$$

and the element

$$\bar{\psi}_{\bar{p}_k} := \prod_{l \in I_k} \left(1 - \frac{\phi_{p_l}}{p_l} \right) = \sum_{n \in \mathcal{N}_k} \frac{\mu(n)}{n} \phi_n \in \mathbb{Z}_{(p_k)}[\phi_{p_l} : l \in I_k].$$

Here $I_k := \{1, \dots, d\} \setminus \{k\}$, \mathcal{N}_k is the monoid of the natural numbers generated by $\bar{p}_k := \mathcal{P} \setminus \{p_k\}$, μ is the Mobius function and the ϕ_{p_l} s are commuting variables. We then may consider the family

$$(\bar{\psi}_{\bar{p}_k} \psi_{p_k}) \in \prod_{k=1}^d \mathcal{O}(\mathcal{J}_{\mathcal{P}}^e(\mathbb{G}_m))^{\widehat{p_k}},$$

where $e = e_1 + \dots + e_d = (1, \dots, 1)$.

Theorem 3.3. *The family $(\bar{\psi}_{\bar{p}_k} \psi_{p_k})$ is a $\delta_{\mathcal{P}}$ -character of order e on \mathbb{G}_m .*

We denote this $\delta_{\mathcal{P}}$ -character by

$$(27) \quad \psi_m^e \in \mathbf{X}_{\mathcal{P}}^e(\mathbb{G}_m).$$

Proof. We notice that

$$(28) \quad \delta_{p_k}(1 + T) = \delta_{p_k} T + C_{p_k}(1, T) \in (T, \delta_{p_k} T) \subset \mathbb{Z}_{p_k}[[T, \delta_{p_k} T]].$$

Then the image of $\bar{\psi}_{\bar{p}_k} \psi_{p_k}$ in $\mathbb{Q}_{p_k}[[\delta_{\mathcal{P}}^i T : i \leq e]]$ is equal the following series (that, by (28), is convergent in the topology given by the maximal ideal of this ring):

$$\begin{aligned} \bar{\psi}_{\bar{p}_k} \left(\sum_{n=1}^{\infty} \frac{(-p_k)^{n-1}}{n} \left(\frac{\delta_{p_k}(1+T)}{(1+T)^{p_k}} \right)^n \right) &= \frac{1}{p_k} \bar{\psi}_{\bar{p}_k} \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{\phi_{p_k}(1+T)}{(1+T)^{p_k}} - 1 \right)^n \right) \\ &= \frac{1}{p_k} \bar{\psi}_{\bar{p}_k} l_{\mathbb{G}_m} \left(\frac{\phi_{p_k}(1+T)}{(1+T)^{p_k}} - 1 \right) \\ &= \frac{1}{p_k} \bar{\psi}_{\bar{p}_k} (\phi_{p_k} - p_k) l_{\mathbb{G}_m}(T) \\ &= - \left(\prod_{l=1}^d \left(1 - \frac{\phi_{p_l}}{p_l} \right) \right) l_{\mathbb{G}_m}(T). \end{aligned}$$

We call ψ_0^e this series.

We have then that ψ_0^e has coefficients in $\mathbb{Q} \cap \mathbb{Z}_{p_k} = \mathbb{Z}_{(p_k)}$, and is the same for all $k = 1, \dots, d$. Hence, ψ_0^e has coefficients in $\mathbb{Z}_{(\mathcal{P})}$ and represents the family $(\bar{\psi}_{\bar{p}_k} \psi_{p_k})$. Further, ψ_0^e satisfies the condition (25) because

$$\begin{aligned} \psi_0^e(\mathcal{G}^e(T_1, T_2)) &= - \left(\prod_{l=1}^d \left(1 - \frac{\phi_{p_l}}{p_l} \right) \right) (l_{\mathbb{G}_m}(T)(\mathcal{G}^0(T_1, T_2))) \\ &= - \left(\prod_{l=1}^d \left(1 - \frac{\phi_{p_l}}{p_l} \right) \right) (l_{\mathbb{G}_m}(T_1) + l_{\mathbb{G}_m}(T_2)) \\ &= \psi_0^e(T_1) + \psi_0^e(T_2). \end{aligned}$$

□

We now prove that ψ_m^e (cf. (27)) generates, in an appropriate sense, the space of all $\delta_{\mathcal{P}}$ -characters of \mathbb{G}_m . We also determine which $\delta_{\mathcal{P}}$ -characters ψ can be analytically continued along any given $\mathbb{Z}_{(\mathcal{P})}$ -point P of \mathbb{G}_m .

Theorem 3.4. *Let ψ be a $\delta_{\mathcal{P}}$ -character of order r on \mathbb{G}_m . Then*

(1) *ψ can be uniquely written as*

$$\psi = \left(\sum_n \rho_n \phi_n \right) \psi_m^e, \quad \rho_n \in \mathbb{Z}_{(\mathcal{P})}.$$

(2) *ψ can be analytically continued along a $\mathbb{Z}_{(\mathcal{P})}$ -point P of \mathbb{G}_m if, and only if, either P is torsion or $\sum_n \rho_n = 0$.*

Consider the *augmentation ideal*

$$\mathbb{Z}_{(\mathcal{P})}[\phi_{\mathcal{P}}]^+ := \left\{ \sum_n \rho_n \phi_n \in \mathbb{Z}_{(\mathcal{P})}[\phi_{\mathcal{P}}] : \sum_n \rho_n = 0 \right\}.$$

Corollary 3.5. *The group of $\delta_{\mathcal{P}}$ -characters $\mathbf{X}_{\mathcal{P}}^{\infty}(\mathbb{G}_m)$ is a free $\mathbb{Z}_{(\mathcal{P})}[\phi_{\mathcal{P}}]$ -module of rank one with basis ψ_m^e . The group of $\delta_{\mathcal{P}}$ -characters in $\mathbf{X}_{\mathcal{P}}^{\infty}(\mathbb{G}_m)$ that can be analytically continued along a given non-torsion point P of \mathbb{G}_m is isomorphic with the augmentation ideal $\mathbb{Z}_{(\mathcal{P})}[\phi_{\mathcal{P}}]^+$ as a $\mathbb{Z}_{(\mathcal{P})}[\phi_{\mathcal{P}}]$ -module. This group is the same for all non-torsion P s.*

Proof of Theorem 3.4. Let ψ_0 be the series representing ψ . We first prove the following:

Claim 1. There is an equality

$$\psi_0 = \left(\sum_{n|\mathcal{P}^r} c_n \phi_n \right) l(T)$$

in $\mathbb{Q}[[\delta_{\mathcal{P}}^i T : i \leq r]]$ with $c_n \in \mathbb{Q}$.

Indeed let $e(T) \in \mathbb{Q}[[T]]$ be the compositional inverse of $l(T)$. Then the series

$$\Theta(\dots, \delta_{\mathcal{P}}^i(T), \dots) := \psi_0(\dots, \delta_{\mathcal{P}}^i(e(T)), \dots)$$

satisfies

$$\Theta(\dots, \delta_{\mathcal{P}}^i(T_1 + T_2), \dots) = \Theta(\dots, \delta_{\mathcal{P}}^i T_1, \dots) + \Theta(\dots, \delta_{\mathcal{P}}^i T_2, \dots).$$

As in the proof of Proposition 3.1, we conclude that $\Theta = \sum c_n \phi_n(T)$, with $c_n \in \mathbb{Q}$, which finishes the proof of the claim.

By Claim 1, by setting $\delta_{\mathcal{P}}^i T = 0$ for $i \neq 0$, we obtain

$$\left(\sum c_n \phi_n\right) \star l(T) \in \mathbb{Z}_{(\mathcal{P})}[[T]],$$

where $\phi_n \star T := T^n$.

Claim 2. If a polynomial $\Lambda = \sum \lambda_n \phi_n \in \mathbb{Q}[\phi_{p_1}, \dots, \phi_{p_s}]$ satisfies

$$\Lambda \star l(T) \in \mathbb{Z}_{p_k}[[T]] \otimes \mathbb{Q}$$

for some $k \in \{1, \dots, s\}$, then Λ is divisible in the ring $\mathbb{Q}[\phi_{p_1}, \dots, \phi_{p_s}]$ by $\phi_{p_k} - p_k$.

Indeed, let us divide Λ by $\phi_{p_k} - p_k$ in $\mathbb{Q}[\phi_{p_1}, \dots, \phi_{p_s}]$ to obtain

$$\Lambda = \left(\sum \alpha_n \phi_n\right)(\phi_{p_k} - p_k) + \sum \beta_n \phi_n,$$

where $\alpha_n, \beta_n \in \mathbb{Q}$ and $\beta_n = 0$ if $p_k | n$. We analyze the remainder term, and prove that it is identically zero by showing that $\beta_n = 0$ for all n .

Since $(\phi_{p_k} - p_k) \star l(T) \in \mathbb{Z}_{p_k}[[T]]$, it follows that $(\sum \beta_n \phi_n) \star l(T) \in \mathbb{Z}_{p_k}[[T]] \otimes \mathbb{Q}$. We may assume $(\sum \beta_n \phi_n) \star l(T) \in \mathbb{Z}_{p_k}[[T]]$. We have that

$$(29) \quad \left(\sum \beta_n \phi_n\right) \star l(T) = \sum_n \sum_m (-1)^{m-1} \beta_n \frac{T^{nm}}{m}.$$

Let us fix integers $n', \nu \geq 1$. By looking at the coefficient of $T^{n'p_k^\nu}$ in (29), we obtain that

$$-\sum_{n|n'} (-1)^{n'/n} \frac{n\beta_n}{n'p_k^\nu} \in \mathbb{Z}_{p_k}$$

because the equality $nm = n'p_k^\nu$ with $n \not\equiv 0 \pmod{p_k}$ implies $m = \mu p_k^\nu$, $\mu \in \mathbb{Z}$, $n\mu = n'$; so $n|n'$ and $m = \frac{n'}{n} p_k^\nu$. Since n is odd for $\beta_n \neq 0$, it follows that

$$\sum_{n|n'} n\beta_n \in p_k^\nu \mathbb{Z}_{p_k},$$

and since this is true for all ν , we obtain that

$$\sum_{n|n'} n\beta_n = 0.$$

But this is true for all n' . By the Mobius' inversion formula, we conclude that $\beta_n = 0$ for all n . This completes the proof of Claim 2.

By Claim 2, it follows that

$$\sum c_n \phi_n = \left(\sum \rho_n \phi_n \right) \prod_{k=1}^d \left(1 - \frac{\phi_{p_k}}{p_k} \right)$$

for some $\rho_n \in \mathbb{Q}$. By induction on n , it is then easy to check that $\rho_n \in \mathbb{Z}_{(\mathcal{P})}$ for all n . This completes the proof of the first assertion (1) in the statement.

In order to prove assertion (2), let $\tau : \mathbb{G}_m \rightarrow \mathbb{G}_m$ be the translation defined by the inverse of P , and let τ^* be the automorphism defined by τ on the various rings of functions. Since $\psi \in \mathcal{O}_{Z, \mathcal{P}}^r(\mathbb{G}_m)$, we have that $\tau^* \psi \in \mathcal{O}_{P, \mathcal{P}}^r(\mathbb{G}_m)$. But if $\psi = (\psi_k)$ then $\tau^* \psi_k = \psi_k - \psi_k(P_k)$ for all k ; cf. the notation in Definition 2.31. Now, if P is torsion or if $\sum_n \rho_n = 0$, it is then clear that $\psi_k(P_k) = 0$. So $\tau^* \psi_k = \psi_k$, hence $\psi \in \mathcal{O}_{P, \mathcal{P}}^r(\mathbb{G}_m)$.

Conversely, let P be non-torsion and given by a number $a \in \mathbb{Z}_{(\mathcal{P})}^\times$. Let $\sum_n \rho_n \neq 0$, and assume $\psi \in \mathcal{O}_{P, \mathcal{P}}^r(\mathbb{G}_m)$. We derive a contradiction. For let $p = p_1$ and $b := a^{p-1} \in 1 + p\mathbb{Z}_{(p)}$, so $b \neq 1$. By Mahler's p -adic analogue of the Hermite-Lindemann theorem [20, 1], we have that $\log(b) \notin \mathbb{Q}$ (where $\log : 1 + p\mathbb{Z}_p \rightarrow p\mathbb{Z}_p$ is the p -adic logarithm). Since $\tau^* \psi \in \mathcal{O}_{P, \mathcal{P}}^r(\mathbb{G}_m)$ and $\psi \in \mathcal{O}_{P, \mathcal{P}}^r(\mathbb{G}_m)$, it follows that $\tau^* \psi - \psi \in \mathcal{O}_{P, \mathcal{P}}^r(\mathbb{G}_m)$. But $\tau^* \psi - \psi = (-\psi_k(a))$ so, in particular, $\psi_1(b) \in \mathbb{Q}$. But

$$\psi_1(b) = -\left(\sum_n \rho_n \right) \cdot \prod_{l=1}^d \left(1 - \frac{1}{p_l} \right) \cdot \log b;$$

cf. the proof of Theorem 3.3. Since $\sum_n \rho_n \neq 0$, it follows that $\log b \in \mathbb{Q}$, reaching the desired contradiction. \square

The next result computes the group of solutions of the $\delta_{\mathcal{P}}$ -character ψ_m^e (27).

Theorem 3.6. *Let A be the $\delta_{\mathcal{P}}$ -ring $\mathbb{Z}_{(\mathcal{P})}[\zeta_m]$ in Example 2.6, and let $\psi_{m,A}^e : \mathbb{G}_m(A_{\mathcal{P}}) = A_{\mathcal{P}}^\times \rightarrow A_{\mathcal{P}}$ be the homomorphism (26) induced by ψ_m^e . Then*

$$\text{Ker } \psi_{m,A}^e = (A_{\mathcal{P}}^\times)_{\text{tors}}.$$

In the statement above, we use the notation Γ_{tors} to denote the torsion group of an abelian group Γ .

Proof. The non-trivial inclusion is “ \subset .” Let us take $Q = (Q_k) \in \text{Ker } \psi_{m,A}^e$ so

$$(30) \quad \bar{\psi}_{\bar{p}_k}(\psi_{p_k}(Q_k)) = 0$$

for all k . Here $Q_k \in A^{\widehat{p_k}} = A^{\widehat{p_{k1}}} \times A^{\widehat{p_{k2}}} \times \dots$, where $p_k = p_{k1} p_{k2} \dots$ is the prime decomposition of $p_k A$. In order to show that Q is torsion, we may replace Q by any of its powers. So we may assume that $Q_k \in 1 + p_k A^{\widehat{p_k}}$ for all k . Then (30) gives

$$\left(\prod_{l=1}^d (\phi_{p_l} - p_l) \right) l_{\mathbb{G}_m}(Q_k - 1) = 0.$$

We claim that the map

$$\begin{aligned} A^{\widehat{p_k}} &\rightarrow A^{\widehat{p_k}} \\ \beta &\mapsto (\phi_{p_l} - p_l) \beta \end{aligned}$$

is injective for all k, l . Using this claim, we conclude that $l_{\mathbb{G}_m}(Q_k - 1) = 0$, which implies that $Q_k = 1$ by the injectivity of $l_{\mathbb{G}_m} : p_k A^{\widehat{p_k}} \rightarrow p_k A^{\widehat{p_k}}$, and finishes the proof of the Theorem.

In order to prove the claim, let us assume that $(\phi_{p_l} - p_l)\beta = 0$. We also have that $\phi_{p_l}^M \beta = \beta$, $M := [\mathbb{Q}(\zeta_m) : \mathbb{Q}]$. Since the polynomials $\phi_{p_l} - p_l, \phi_{p_l}^M - 1 \in \mathbb{Q}[\phi_{p_l}]$ are coprime, it follows that $\beta = 0$, as desired. \square

3.3. Elliptic curves. Let E be an elliptic curve over \mathbb{Q} . We assume that E has minimal model over \mathbb{Z} given by

$$y^2 + c_1xy + c_3y = x^3 + c_2x^2 + c_4x + c_6.$$

We assume further that the discriminant of E is not divisible by any of the primes in \mathcal{P} , and view E as an elliptic curve (smooth projective group scheme) over $\mathbb{Z}_{(\mathcal{P})}$. Then the zero section is uniform, with uniform coordinate $T = \frac{x}{2y}$. Let $\mathcal{G}^0 \in \mathbb{Z}_{(\mathcal{P})}[[T_1, T_2]]$ be the formal group law attached to E with respect to T , and let $l_E \in \mathbb{Q}[[T]]$ be the logarithm of \mathcal{G}^0 . So

$$l_E(T) = \sum b_n \frac{T^n}{n},$$

where

$$\frac{dx}{2y + c_1x + c_3} = \sum b_n T^{n-1} dT.$$

Let $a_{p_k} \in \mathbb{Z}$ be the trace of the p_k -power Frobenius on the reduction mod p_k of E . By (20) and [7], Theorem 1.10, there exists

$$\psi_{p_k} = \psi_{p_k}^2 \in \mathcal{O}(\mathcal{J}_{\{p_k\}}^2(E)^{\widehat{p_k}})$$

whose image via

$$\mathcal{O}(\mathcal{J}_{\{p_k\}}^2(E)^{\widehat{p_k}}) \rightarrow \mathbb{Q}_{p_k}[[T, \delta_{p_k}T, \delta_{p_k}^2T]]$$

equals

$$\frac{1}{p_k}(\phi_{p_k}^2 - a_{p_k}\phi_{p_k} + p_k)l_E(T).$$

On the other hand, we may consider the element

$$\bar{\psi}_{\bar{p}_k} := \prod_{l \in I_k} \left(1 - a_{p_l} \frac{\phi_{p_l}}{p_l} + p_l \left(\frac{\phi_{p_l}}{p_l} \right)^2 \right) \in \mathbb{Z}_{p_k}[\phi_{p_l} : l \in I_k],$$

where $I_k = \{1, \dots, d\} \setminus \{k\}$. Let us still denote by ψ_{p_k} the image of ψ_{p_k} via the homomorphism

$$\mathcal{O}(\mathcal{J}_{\{p_k\}}^2(E)^{\widehat{p_k}}) \rightarrow \mathcal{O}(\mathcal{J}_{\mathcal{P}}^{2e_k}(E)^{\widehat{p_k}})$$

induced by (20). Then, we may consider the family

$$(\bar{\psi}_{\bar{p}_k} \psi_{p_k}) \in \prod_{k=1}^d \mathcal{O}(\mathcal{J}_{\mathcal{P}}^{2e}(E)^{\widehat{p_k}}).$$

Theorem 3.7. *The family $(\bar{\psi}_{\bar{p}_k} \psi_{p_k})$ is a $\delta_{\mathcal{P}}$ -character of order $2e$ on E .*

We denote this $\delta_{\mathcal{P}}$ -character by

$$(31) \quad \psi_E^{2e} \in \mathbf{X}_{\mathcal{P}}^{2e}(E).$$

Proof. As in the proof of Theorem 3.3, the image of $\bar{\psi}_{\bar{p}_k} \psi_{p_k}$ in the ring

$$\mathbb{Q}_{p_k}[[\delta_{\mathcal{P}}^i T ; i \leq 2e]]$$

is equal to the series

$$\psi_0^{2e} := \left[\prod_{l=1}^d \left(1 - a_{p_l} \frac{\phi_{p_l}}{p_l} + p_l \left(\frac{\phi_{p_l}}{p_l} \right)^2 \right) \right] l_E(T),$$

which is independent of k and has coefficients in $\mathbb{Z}_{(p_k)}$. Also, as in loc.cit. we have that

$$\psi_0^{2e}(\mathcal{G}^{2e}(T_1, T_2)) = \psi_0^{2e}(T_1) + \psi_0^{2e}(T_2).$$

□

As in the case of the multiplicative group \mathbb{G}_m , we now have the following.

Theorem 3.8. *Let us assume that the elliptic curve E has ordinary (good) reduction at all the primes in \mathcal{P} . Let ψ be a $\delta_{\mathcal{P}}$ -character of order r on E . Then*

(1) *ψ can be uniquely written as*

$$\psi = \left(\sum_n \rho_n \phi_n \right) \psi_E^{2e}, \quad \rho_n \in \mathbb{Z}_{(\mathcal{P})}.$$

(2) *ψ can be analytically continued along a $\mathbb{Z}_{(\mathcal{P})}$ -point P of E if, and only if, either P is torsion or $\sum_n \rho_n = 0$.*

Corollary 3.9. *The group of $\delta_{\mathcal{P}}$ -characters $\mathbf{X}_{\mathcal{P}}^{\infty}(E)$ is a free $\mathbb{Z}_{(\mathcal{P})}[\phi_{\mathcal{P}}]$ -module of rank one with basis ψ_E^{2e} . The group of $\delta_{\mathcal{P}}$ -characters in $\mathbf{X}_{\mathcal{P}}^{\infty}(E)$ that can be analytically continued along a given non-torsion point P of E is isomorphic with the augmentation ideal $\mathbb{Z}_{(\mathcal{P})}[\phi_{\mathcal{P}}]^+$ as a $\mathbb{Z}_{(\mathcal{P})}[\phi_{\mathcal{P}}]$ -module. This group is the same for all non-torsion P s.*

Proof of Theorem 3.8. Proceeding as in the proof of Theorem 3.4, if ψ_0 is the series representing ψ , then the following claim holds:

Claim 1. There is an equality

$$\psi_0 = \left(\sum_{n|\mathcal{P}^r} c_n \phi_n \right) l_E(T)$$

in $\mathbb{Q}[[\delta_{\mathcal{P}}^i T ; i \leq r]]$ with $c_n \in \mathbb{Q}$.

By Claim 1, we obtain that

$$\left(\sum c_n \phi_n \right) \star l_E(T) \in \mathbb{Z}_{(\mathcal{P})}[[T]].$$

We also have the following:

Claim 2. If a polynomial $\Lambda = \sum \lambda_n \phi_n \in \mathbb{Q}[\phi_{p_1}, \dots, \phi_{p_d}]$ satisfies

$$\Lambda \star l_E(T) \in \mathbb{Z}_{p_k}[[T]] \otimes \mathbb{Q}$$

for some $k \in \{1, \dots, d\}$, then Λ is divisible in the ring $\mathbb{Q}[\phi_{p_1}, \dots, \phi_{p_d}]$ by $\phi_{p_k}^2 - a_{p_k} \phi_{p_k} + p_k$.

Indeed, by the fact that E has ordinary reduction at p_k , we may consider the unique root $u_k p_k$ of $x^2 - a_{p_k} x + p_k$ in $p_k \mathbb{Z}_{p_k}$. Since this root is not in \mathbb{Q} , it is enough to prove that Λ is divisible in $\mathbb{Q}_{p_k}[\phi_{p_1}, \dots, \phi_{p_d}]$ by $\phi_{p_k} - u_k p_k$. Let

$$L(E, s) = \sum a_n n^{-s}$$

be the L -function of E . Let

$$f_E(T) := \sum a_n \frac{T^n}{n}.$$

As a consequence of the Euler product structure of $L(E, s)$, it follows that

$$(32) \quad (\phi_{p_k} - u_k p_k) \star f_E(T) \in \mathbb{Z}_{p_k}[[T]],$$

cf. [14], p. 441. By the Honda-Hill Theorem [15], p. 76, there exists a series $h(T) = T + \dots \in \mathbb{Z}[[T]]$ such that $f_E(T) = l_E(h(T))$. By the Functional Equation Lemma [15], p.74,

$$(\phi_{p_k} - u_k p_k) \star l_E(T) \in \mathbb{Z}_{p_k}[[T]].$$

Now, as in the proof of Theorem 3.4, we divide Λ by $\phi_{p_k} - u_k p_k$ in the ring $\mathbb{Q}_{p_k}[\phi_{p_1}, \dots, \phi_{p_d}]$, to obtain

$$\Lambda = \left(\sum \alpha_n \phi_n \right) (\phi_{p_k} - u_k p_k) + \sum \beta_n \phi_n$$

with $\alpha_n, \beta_n \in \mathbb{Q}_{p_k}$, $\beta_n = 0$ for $p_k | n$, and proceed to analyze the remainder.

By (32), we get $(\sum \beta_n \phi_n) \star l_E(T) \in \mathbb{Z}_{p_k}[[T]] \otimes \mathbb{Q}$. We claim that $\beta_n = 0$ for all n . We may assume $(\sum \beta_n \phi_n) \star l_E(T) \in \mathbb{Z}_{p_k}[[T]]$. We have

$$(33) \quad \left(\sum \beta_n \phi_n \right) \star l_E(T) = \sum_n \sum_m a_m \beta_n \frac{T^{nm}}{m}.$$

Fix integers $n', \nu \geq 1$. As in the proof of Theorem 3.4, we get

$$\sum_{n|n'} a_{p_k^\nu n'/n} n \beta_n \in p_k^\nu \mathbb{Z}_{p_k}.$$

Assume $n|n'$ and $n' \not\equiv 0 \pmod{p_k}$. Then $a_{p_k^\nu n'/n} = a_{p_k^\nu} a_{n'/n}$. Since E has ordinary reduction at p_k , we have that $a_{p_k^\nu} \not\equiv 0 \pmod{p_k}$. We conclude that

$$\sum_{n|n'} a_{n'/n} n \beta_n \in p_k^\nu \mathbb{Z}_{p_k}.$$

Since ν is arbitrary, we get

$$\sum_{n|n'} a_{n'/n} n \beta_n = 0.$$

Since this is true for any $n' \not\equiv 0 \pmod{p_k}$, we get that the following product of formal Dirichlet series with coefficients in \mathbb{Q}_{p_k} vanishes:

$$\left(\sum_{n \not\equiv 0(p_k)} n \beta_n n^{-s} \right) \left(\sum_{n \not\equiv 0(p_k)} a_n n^{-s} \right) = 0.$$

We conclude that $\beta_n = 0$ for all n , which completes the proof of Claim 2.

From this point on, we may proceed as in the proof of Theorem 3.4 to derive assertion (1). Assertion (2) can be proved exactly as in the case of Theorem 3.4. Instead of Mahler's theorem [20], we now need to use that $l_E(c) \notin \mathbb{Q}$ for any $0 \neq c \in p\mathbb{Z}_{(p)}$, and the latter follows by Bertrand's elliptic p -adic analogue of the Hermite-Lindemann theorem [1]. \square

In the next result, we compute the group of solutions of the $\delta_{\mathcal{P}}$ -character ψ_E^{2e} given by (31).

Theorem 3.10. *Let A be the $\delta_{\mathcal{P}}$ -ring $\mathbb{Z}_{(\mathcal{P})}[\zeta_m]$ in Example 2.6, and let $\psi_{E,A}^{2e} : E(A_{\mathcal{P}}) \rightarrow A_{\mathcal{P}}$ be the homomorphism (26) induced by ψ_E^{2e} . Then*

$$\text{Ker } \psi_{E,A}^{2e} = E(A_{\mathcal{P}})_{tors}.$$

This can be viewed as a version for several primes of the arithmetic analogue [7] of Manin's Theorem of the Kernel [21].

Proof. We use the same argument as in the proof of Theorem 3.6. We use here the fact that the polynomials $\phi_{p_l}^2 - a_{p_l}\phi_{p_l} + p_l, \phi_{p_l}^M - 1 \in \mathbb{Q}[\phi_{p_l}]$ are coprime. \square

4. FINAL REMARKS AND QUESTIONS

We see an emerging pattern from the construction of $\delta_{\mathcal{P}}$ -characters in the previous section. It can be best explained via the following:

Definition 4.1. Let X be a quasi-projective smooth scheme over $\mathbb{Z}_{(\mathcal{P})}$ with irreducible geometric fibers, and let $\mathcal{P} = \{p_1, \dots, p_d\}$ be a finite set of primes not dividing N . Let P be a uniform point of X . A $\delta_{\mathcal{P}}$ -function

$$f = (f_k) \in \mathcal{O}_{P, \mathcal{P}}^r(X)$$

will be said to have a *Dirac decomposition* if there exists a family

$$(\eta_{p_k}) \in \prod_{k=1}^d \mathcal{O}(\mathcal{J}_{\{p_k\}}^s(X)^{\widehat{p_k}}),$$

$s \in \mathbb{Z}_+$, and a family

$$(\bar{\eta}_{\bar{p}_k}) \in \prod_{k=1}^d \mathbb{Z}_{p_k}[\phi_l : l \in I_k],$$

$I_k := \{1, \dots, d\} \setminus \{p_k\}$, $\bar{p}_k = \mathcal{P} \setminus p_k$, such that $f_k = \bar{\eta}_{\bar{p}_k} \eta_{p_k}$ for all k .

Then the following remarks are in order:

Remark 4.2. Any $\delta_{\mathcal{P}}$ -character of $\mathbb{G}_a, \mathbb{G}_m, E$ (where E is an elliptic curve over $\mathbb{Z}_{(\mathcal{P})}$) has a Dirac decomposition. \square

Remark 4.3. Let X be a projective curve over $\mathbb{Z}_{(\mathcal{P})}$ of genus ≥ 2 with a uniform point P . Can we always find $\delta_{\mathcal{P}}$ -functions on X not belonging to $\mathbb{Z}_{(\mathcal{P})}$? In other words, do we have $\mathcal{O}_{P, \mathcal{P}}^\infty(X) \neq \mathbb{Z}_{(\mathcal{P})}$? (Note that by [6] we always have $\mathcal{O}(\mathcal{J}_{\{p_k\}}^1(X)^{\widehat{p_k}}) \neq \mathbb{Z}_{p_k}$ for each k .) Can we always find a $\delta_{\mathcal{P}}$ -function in $\mathcal{O}_{P, \mathcal{P}}^\infty(X) \setminus \mathbb{Z}_{(\mathcal{P})}$ which has a Dirac decomposition? By Theorem 3.7, the answer to the last question is positive if X has a morphism to an elliptic curve over $\mathbb{Z}_{(\mathcal{P})}$; by Eichler-Shimura, this is always the case for appropriate \mathcal{P} s and X a modular curve of sufficiently high level. \square

Remark 4.4. Let $X = \text{Spec } \mathbb{Z}[1/6][a_4, a_6, (4a_4^3 + 27a_6^2)^{-1}]$. The theory of differential modular forms in the ODE setting [8, 9] is, essentially, the study of some remarkable elements of the rings $\mathcal{O}(\mathcal{J}_{\{p_k\}}^n(X)^{\widehat{p_k}})$ satisfying certain homogeneity properties. We may ask, in the arithmetic PDE setting, for the existence and structure of $\delta_{\mathcal{P}}$ -modular forms defined as $\delta_{\mathcal{P}}$ -functions on X with appropriate homogeneity properties. Some such forms can be constructed via Dirac decompositions, using Remark 4.3 above. \square

Remark 4.5. The construction of non-trivial $\delta_{\mathcal{P}}$ -characters on one dimensional groups via Dirac decompositions is a consequence of the fact that the “ring of symbols,” $\mathbb{Z}_{(\mathcal{P})}[\phi_{\mathcal{P}}]$, is a commutative integral domain. Indeed, the key property used to construct Dirac decompositions is that

- (*) For any collection of non-zero elements $\Theta_1, \dots, \Theta_d$ in $\mathbb{Z}_{(\mathcal{P})}[\phi_{\mathcal{P}}]$, there exist elements $\Lambda_1, \dots, \Lambda_d$ in $\mathbb{Z}_{(\mathcal{P})}[\phi_{\mathcal{P}}]$ such that $\Lambda_1 \Theta_1 = \dots = \Lambda_d \Theta_d \neq 0$.

This property is trivially true in any commutative integral domain.

Now, we can ask for a generalization of the theory of the present paper in two possible directions:

- (a) arithmetic PDEs in $0 + d$ independent variables ($d \geq 2$), and $d_3 \geq 2$ dependent variables (corresponding to commutative algebraic groups of dimension ≥ 2);
- (b) arithmetic PDEs in $1 + d$ independent variables ($d \geq 2$) and $d_3 = 1$ dependent variables (corresponding to $\mathbb{G}_a, \mathbb{G}_m, E$).

However, in both cases, (a) and (b), the “rings of symbols” are non-commutative and, in case (a), the corresponding ring has zero divisors. Indeed, in case (a) the “ring of symbols” turns out to be the ring of $d_3 \times d_3$ matrices with coefficients in $\mathbb{Z}_{(\mathcal{P})}[\phi_{\mathcal{P}}]$. In case (b), the “ring of symbols” turns out to be the ring $\mathbb{Z}_{(\mathcal{P})}[\delta_q, \phi_{\mathcal{P}}]$ generated by variables $\delta_q, \phi_{p_1}, \dots, \phi_{p_d}$ with the ϕ_{p_k} s commuting with each other, and with $\delta_q \phi_{p_k} = p_k \cdot \phi_{p_k} \delta_q$ for all k ; cf. [10] for the case $d = 1$.

In both cases, (a) and (b), property (*) fails for the corresponding rings of symbols (and for the elements in those rings that are relevant to our problem) so the construction of characters with Dirac decompositions fails as it is. An interesting problem would then be to determine all the characters in cases (a) and (b) respectively. \square

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DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF NEW MEXICO, ALBUQUERQUE,
NM 87131

E-mail addresses: buium@math.unm.edu, santiago@math.unm.edu